Teaching Mathematical Problem Solving:
An Analysis of an Emergent Classroom Community

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Introduction[1]

> Toward the end of the semester I assigned the following. . . . As usual, the class broke into groups to work on the problem. One group became the staunch defenders of one conjecture, while a second group lobbied for another. The two groups argued somewhat heatedly, with the rest of the class following the discussion. Finally, one group prevailed, on what struck me as solid mathematical grounds. As is my habit, I did not reveal this but made my usual comment: "OK, you seem to have done as much with this as you can. Shall I try to pull things together?" One of the students replied, "Don't bother. We got it." The class agreed. (Schoenfeld, 1994, pp. 63-64)

Two main goals of Alan Schoenfeld's problem solving course are illustrated by this anecdote: That the class function as a "mathematical community" advancing and defending conjectures and proofs on mathematical grounds, and that the locus of authority be the "mathematical community," not the teacher (Schoenfeld, 1994, p. 65). Such incidents are not common in undergraduate mathematics classes, whether they are composed of elite mathematics majors or students struggling through their first calculus course. All too often students seem passive, disengaged, and untroubled by contradictions in their work.

After twelve or more years of schooling, most undergraduates have well-developed expectations for mathematics classes based on their many experiences of listening, taking notes, and learning procedures to solve standard problems. In order to establish his "classroom community," Schoenfeld must convey his nonstandard expectations for behavior and at the same time convince his students that he knows what he's talking about, that his course is of value, and that heuristics as well as formalism are essential parts of doing mathematics. In other words, Schoenfeld must renegotiate the "didactic contract" (Brousseau, 1986) with his students. This contract includes, among other things, teachers' and students' understandings of what is to be expected in the classroom: "What assistance can the students reasonably expect from the teacher; what assistance can the students seek from each other; what level of explanation is the teacher obliged to provide;

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[1]This paper is the product of a long and enjoyable collaboration that began in 1990, in Berkeley, California, and continued over six years and four continents (thanks to e-mail). Each major section was individually developed and thus has a single author, though all of us critiqued each section. The *Introduction* and the *Concluding Discussion* reflect our shared views, and each of us had some part in writing them. However, Abraham Arcavi, Luciano Meira, and Jack Smith would like to thank Cathy Kessel who composed these sections with unusual editorial care and wisdom.

The authors thank the editors, Ed Dubinsky and Jim Kaput; the reviewers, Barbara Pence, Beth Warren, and one anonymous reviewer; and members of the Functions Group, Ilana Horn, Andrew Iszak, Sue Magidson, and Natasha Speer, for their help in improving the successive versions of this article.

We owe special thanks to Alan Schoenfeld. This article would not have been possible without his cooperation. It is not easy to be the subject of any analysis, let alone one so prolonged. Schoenfeld not only cooperated with us, but did so with grace, tolerance, and generosity.

what questions can the teacher reasonably ask; what form of response will be considered satisfactory" (Clarke, 1995 April).

Schoenfeld's situation is an instance of a more general problem: If a course differs radically from students' previous courses, how can its instructor convince students that the course is worthwhile and convey expectations for classroom behavior? This problem is often encountered by those who teach first-year undergraduate courses. It is also faced by teachers of reformed calculus courses (Cipra, 1995; Culotta, 1992; University of Michigan, 1993). Students are uncomfortable with changed teaching and expectations. At some universities this discomfort has brought calculus reform to a halt (Cipra, 1995, p. 19).

Like Schoenfeld, some instructors have solved the problem of how to establish desired classroom norms. Others have noted it, but not yet developed a solution. Those new to teaching may not be aware the problem exists. We think that our description of Schoenfeld's solution will be of interest to people in each of these categories and present this account, not as a prescription to be followed, but as an example that might illuminate aspects of the difficult task of mathematics teaching.

We also hope to suggest an analytic way in which to talk about teaching and attempt to make salient considerations that are sometimes overlooked. Instead of describing selected incidents from the class or characterizing Schoenfeld's "teaching style," we focus on making sense of all the teaching actions at the beginning of the course, describing actions in detail as well as providing a rationale. Our account offers a method of description as well as the description itself.

In the fall of 1990, Schoenfeld taught his undergraduate course Mathematical Problem Solving in the mathematics department of the University of California at Berkeley. The course is designed to provide students with an introduction to what it means to think mathematically. It is an elective course. The prerequisite is one semester of calculus or consent of instructor.

In order to build an empirical base for his own in-depth analysis of the course, Schoenfeld asked the first author (who had already attended this class at Berkeley in 1987) to videotape each of the 29 two-hour class meetings. The first author attended and videotaped each class session, and the other three authors also attended numerous class sessions. All four authors were members of Schoenfeld's research group[2] at the time, and they became interested in the same question: How does Schoenfeld create a classroom community of problem solvers in which undergraduates learn to think and do mathematics?

This question shapes our analysis. We focus on the initial stages of the community, using the videotape records of the first two class sessions as the principal data. Other data include the remaining 27 videotapes, interviews with students after the class ended, audiotaped discussions with Schoenfeld, and Schoenfeld's writings. To understand the emergence of this community, we examine how Schoenfeld understood and made the connections between his mathematics classroom and the professional community of practicing mathematicians; how he stated and enacted his expectations for student participation; how he introduced students to desired forms of mathematical discourse and activity; and how he introduced heuristics in the context of specific problems.

At this point it is important to delimit the scope of this paper.

- It is not our intention to describe, compare, or ignore the design and implementation of successful mathematics classroom practices. We believe that there are many such practices, but we decided to concentrate on this one because we were fortunate enough to observe, analyze, and discuss it in depth. We hope this example will encourage other researchers to offer similarly detailed accounts of mathematics teaching and learning.

- We do not analyze the success of the course. This has been documented: Students learn to use heuristics successfully (Schoenfeld, 1982; 1985) and the class becomes a "mathematical community" (Schoenfeld 1989a; 1991; 1992a; 1992b; 1994).

- We do not provide an analysis of how the classroom evolved over the semester, how students participated, how they learned, how they changed from being rather silent to being very involved. We decided to focus on Schoenfeld's teaching at the very beginning for two main reasons: first, the analyses suggest some very interesting and counter-intuitive findings

[2]Schoenfeld's research group, the Functions Group at the University of California at Berkeley, has been involved since 1985 in a series of research and development studies of mathematical teaching and learning.

about the initial stages of such a class; and second, if continued through the remaining 54 hours of tape, the analyses would result in a long book rather than a long article.

As one might expect, data in the form of videotapes and interviews require methods of analysis different from those used to examine data that are more uniform and manageable. Our analytic method, often known as microanalysis, has roots in cognitive science and ethnography. Schoenfeld, Smith, and Arcavi (1993) describe it as striving "for explanations that are both locally and globally consistent, acounting for as much observed detail as possible and not contradicting any other related explanations." (For general discussion of this method, see Schoenfeld, 1988; Schoenfeld, Smith, & Arcavi, 1993; Schoenfeld, 1992b. For discussion of related methods in the analysis of teaching, see Schoenfeld et al., 1992; Schoenfeld, Minstrell, & van Zee, 1996 April.)

After our initial collaborative analysis of the first two videotapes, each author selected instructional segments from those tapes to analyze in detail. Each analysis was discussed with the other authors. The results are the four main sections of this article. Though we share a common interest in mathematics education, our backgrounds are different enough for us to provide a multilayered view of the class. Arcavi has a Master's degree in mathematics and a Ph.D. in mathematics education. He has taught secondary school mathematics for 10 years, in teacher education programs for 10 years, and has been involved in curriculum development and research on mathematics teaching and learning for the past 15 years. Meira has a Master's degree in cognitive psychology and a Ph.D. in mathematics education. He has teaching experience at the elementary, secondary and tertiary levels, and has been involved in research on mathematics teaching and learning for the past ten years. Smith taught upper elementary, middle, and high school mathematics for 6 years, with a B.A. in mathematics before obtaining a Ph.D. in educational psychology. His research centers on detailed analyses of student understanding and learning of precollege mathematics. Kessel has a Ph.D. in mathematics, taught lower and upper division undergraduate courses as a teaching assistant, lecturer, and assistant professor for 14 years, and works as a researcher in mathematics education.

In the first of these sections, *An Overview of the Problem Solving Course*, Arcavi provides a general description of the goals, curriculum, and pedagogy of the course, as well as some background on the 1990 class. In the second section

Presenting and Doing Mathematics: An Introduction to Heuristics, Meira analyzes how mathematical problem solving was first discussed and enacted. Central to this analysis are the distinctions among *doing* mathematics, *presenting* mathematics, and *presenting how to do* mathematics, which Meira uses to examine the complex relationships among professional mathematics, school mathematics, and Schoenfeld's course. Smith's section *Making the Case for Heuristics: Authority and Direction in the Inscribed Square*, focuses on the solution to the second problem of the course, analyzing how heuristics were introduced and how the students' work with them was managed. His emphasis on the role of Schoenfeld's leadership and authority in the early days of the class, shows how complex and counterintuitive the initiation of a classroom community can be. In *Practicing Mathematical Communication: Using Heuristics with the Magic Square*, Kessel discusses the third problem of the course, focusing on Schoenfeld's use of traditional and non-traditional mathematical language and discourse. Smith's and Kessel's sections discuss different aspects of the complex interplay between teacher authority and communal judgment, and between traditional and nontraditional elements of Schoenfeld's pedagogy. In the *Concluding Discussion* we summarize what we consider to be the most important issues arising from these analyses and offer some implications.

An overview of the problem solving course

Abraham Arcavi

This section describes Schoenfeld's problem solving course, providing the context and background for the analysis of the following sections. The description includes his professional background, his goals for the course, the basic characteristics of the "classroom culture" he wants to create in order to achieve them, the curriculum and pedagogy, and some details about the 1990 class. The description is based on our observations of the 1987 and 1990 classes, on personal dialogues with Schoenfeld, and his written accounts (Schoenfeld, 1983; 1985; 1988; 1989a; 1991; 1994).

Background and goals

Schoenfeld's conceptualization, design and teaching of his course draw upon three distinct but related fields: mathematics, cognitive science, and college teaching. As a member of the mathematical community, he has examined the nature of his own mathematical activity and the practices of the discipline itself (Schoenfeld, 1983; 1985; 1991; 1994). His participation in that professional practice is directly reflected in his goals for his course. As a cognitive scientist, he has conducted extensive research on the nature of mathematical problem solving and thinking (Schoenfeld, 1985; 1987; 1992; Schoenfeld, Smith, & Arcavi, 1993). This research has provided detailed models of successful (and unsuccessful) problem solving which have directly influenced his teaching practice. As a teacher of college mathematics, he designed the problem solving course, taught, evaluated, and revised it over a period of more than 15 years (Schoenfeld, 1983; 1985; 1991; 1994).

Though most students who take the course are among the successes of the mathematics education system, they begin the course with very different expectations and practices from those envisioned by Schoenfeld. In the area of problem solving, students' past experiences in mathematics consist mostly of generating "the answer" to problems by applying procedures for manipulating numerical and symbolic expressions. They have learned to view the professor (and/or the textbook) as the sole authorities in the classroom and to defer to this external judgment on most issues. They have developed an ability to "master" facts and procedures for exams which are the accepted evidence of their mathematical competence. However, given problems out of context they may well not know when to apply those facts and procedures. Moreover, as Schoenfeld (1994, p. 43) notes, "For most of them, doing mathematics has meant studying material and working tasks set by others, with little or no opportunity for invention or sustained investigations."

A major goal of Schoenfeld's problem solving course is to provide his students with the opportunity to engage in doing mathematics by creating and supporting a "classroom culture" in which students *can* solve problems given out of context, judge the validity of their solutions without appealing to an external authority, and have the opportunity for invention and sustained investigations. These aspects of the course are consistent with those of the mathematical community.

Characteristics of the "classroom culture"

The task of creating and nurturing the "mathematics classroom culture" in which students will have the experience of doing mathematics, has the following characteristics.

Development of a mathematical point of view. According to Schoenfeld, doing mathematics is more than acquiring the primary tools (e.g., facts and procedures) and deploying them thoughtfully in solving problems. It involves looking at the world with "mathematical spectacles" in a wide variety of problem situations—using mathematics to symbolize, abstract, model, prove or disprove conjectures; perceiving connections across problems and results; and creating knowledge that is new to oneself or to the community. The search for and discussion of solutions to problems is not the only focus of the activity. Problems should also serve as springboards for generalization (or specialization), to establish connections between mathematical domains, to reveal mathematical structure, and to pose new problems.

Emphasis on processes as well as on results. While emphasizing and honoring results that are new to the class, Schoenfeld gives greater priority to the reasoning processes that generated the results. Results that students cannot explain, regardless of their correctness, power, or appeal, are not valued. Tricks, results proven elsewhere ("we proved in Math 127 that . . . "), and "rabbits pulled out of hats," are dismissed in favor of presentations of accessible and non-technical mathematical arguments that presume only what is known by all members of the community. Explanations of how ideas are generated are highly valued, even when they do not produce solutions.

Communication. Students' mathematical activity takes place in an inherently social milieu, where they work as individuals, as members of small groups and as participants in whole-class activities. The classroom setting encourages and supports various levels of oral and written mathematical communication: from expressions of raw ideas, suggestions, intuitions, or insights to top-level descriptions of mathematical arguments and also final, polished, and airtight mathematical presentations. Students are encouraged to evaluate, question, and criticize each other's suggestions and work, in both small-group and whole-class activities. Schoenfeld plays a strong role in shaping this communication, ensuring that students criticize the mathematics rather than each other (Schoenfeld, 1994).

Leadership and authority. As the established leader of this community, Schoenfeld's task requires both a detailed overall design and a continual day-to-day shaping. He sets the top-level goals, selects the initial problems, directs students' work on those problems, models desirable mathematical actions and dispositions, and consults with individuals and groups of students. He states his goals openly and explicitly relates them to specific teaching actions and decisions. However, his intention is to gradually transfer the locus of authority and community leadership from himself to the students as they become more comfortable with him and the class. He starts by deferring to students in evaluating the validity of proposed solutions. He asks them to question and challenge any hidden or explicit parts of mathematical arguments that are unconvincing, unclear, or based on implicit knowledge not shared by the whole class. Schoenfeld leads the class towards assuming responsibility for safeguarding standards, for what can be accepted as "basic" shared knowledge and for the completeness, coherence, and conviction of mathematical arguments. He does so by asking very direct questions and by providing initial modeling of desirable actions.

Though he expects students to play a substantial role in setting the mathematical agenda, in presenting their thinking, and in evaluating each others' arguments, he reserves the right to encourage the class in certain directions and not others. He describes this as follows:

> I know what fruitful directions are that students are likely to [engage in], or can be nudged into, and on the basis of my general sense of what's mathematically valuable, I'm going to try, without letting the students know it, to nudge the conversation in the direction of things which I consider important, giving enough latitude to go where they think it's right. It's clear that that works in the sense that, in a number of classes they've discovered mathematics which I didn't know, it was good mathematics . . . on the other hand, I am nudging away from things that are frivolous, not necessarily dead ends (because dead ends can be profitable), and I try to do that in a way which is not terribly overt, but someone who really understands the mathematics and the goals for my class can clearly pick that up. (From an audiotaped conversation with Schoenfeld about his course, May, 1991)

Reflective mathematical practice. Learning to solve problems and think mathematically requires continuous reflection on the nature of that activity. Questions that Schoenfeld first asks of students almost routinely, are intended to play a central role in developing that reflective capability. For example, Schoenfeld has shown that skillful mathematical problem solving includes the development of a critical attitude toward mathematical argument: "Is this airtight?," "Does it convince me, a friend, an enemy?" (Mason, Burton, & Stacey, 1982; Schoenfeld, 1994), "Am I done with this problem?" Other questions help to develop the mathematical point of view: "How could this have been done in another way?," "How can this result be generalized?," "Is this result similar to another we have seen?" and so on.

Later in the course, Schoenfeld also devotes time to develop what he calls "executive control of students' solution attempts." Briefly stated, control is "a category of behavior [which] deals with the way individuals use the information potentially at their disposal. It focuses on major decisions about what to do in a problem, decisions that in and of themselves may 'make or break' an attempt to solve a problem. Behaviors of interest include making plans, selecting goals and subgoals, monitoring and assessing solutions as they evolve, and revising or abandoning plans when the assessments indicate that such actions should be taken" (Schoenfeld, 1985, p. 27). Schoenfeld nurtures this behavior by asking students the following questions while they work: (1) "What are you doing?," (2) "Why are you doing it?," and (3) "How does it help you?" (Schoenfeld, 1985; 1988;

1992a). Schoenfeld suggests that these kinds of questions are slowly internalized and become an integral part of the students' doing mathematics.

Curriculum

The curriculum of the course consists of a collection of carefully chosen problems, drawn from a wide variety of mathematical domains—including number theory, Euclidean constructions, cryptarithmetic, calculus, algebra, and probability. Problems are presented to students on sheets distributed in class. In general, each problem sheet does not have more than one or two problems from the same mathematical domain.

This aspect of the curriculum addresses one of the main course goals: That students learn how to solve problems out of context. In traditional courses, problems and exercises are often sequenced in such a way that students can easily find solution techniques. Thus problems are perceived as mere opportunities to exercise a pre-established and known technique. Schoenfeld deliberately chooses not to sequence problems from the same mathematical domain consecutively. On the contrary, whenever he feels a technique or a solution strategy is understood, he changes the type of problem, even giving examples in which the thoughtless application of a recently "mastered" technique can lead to error or nowhere. Thus the sequencing of the problems is consonant with his intention to teach students to approach problems as professionals do, namely without having explicit cues about the techniques to be used. Because of that, to a casual observer, it may seem that the design of the course has discontinuities and lacks coherence: a result reached in one class session may not be recalled or invoked until three or four sessions later when the result is relevant, useful, or connected with the issues discussed.

Since Schoenfeld is not constrained to "covering" a predetermined amount of content, he can afford to allocate time flexibly so that work and discussion can yield the maximum mathematical profit. Problems which can be solved in minutes with traditional "show and tell" teaching are worked on and discussed as long as they have mathematical substance, fulfilling one of the main course goals: That students have the opportunity for invention and sustained investigations.

What are problems and how are they chosen? Are there any criteria for good problems? Schoenfeld regards problems as demanding, non-routine and interesting mathematical tasks, which students want and like to solve, and for which they lack readily accessible means to achieve a solution (Schoenfeld, 1985; 1989a). Problems selected for the course must satisfy five main criteria (Schoenfeld, 1994).

- Without being trivial, problems should be accessible to a wide range of students on the basis of their prior knowledge, and should not require a lot of machinery and/or vocabulary.
- Problems must be solvable, or at least approachable, in more than one way. Alternative solution paths can illustrate the richness of the mathematics, and may reveal connections among different areas of mathematics.
- Problems should illustrate important mathematical ideas, either in terms of the content or the solution strategies.
- Problem solutions should be constructible without tricks.
- Problems should serve as first steps towards mathematical explorations, they should be extensible and generalizable; namely, when solved, they can serve as springboards for further explorations and problem posing.

A main topic in Schoenfeld's curriculum is, as already implied, *heuristics*—"rules of thumb for successful problem solving, general suggestions that help an individual to understand a problem better or to make progress towards its solution" (Schoenfeld, 1985, p. 23). Commonly used heuristics include: exploiting analogies, examining special cases, arguing by contradiction, working backwards, decomposing and recombining, exploiting related problems, generalizing and specializing, and relaxing conditions in the problem (see Pólya, 1973 for a more complete list). The rationale for teaching heuristics is clear: expert problem solvers develop and rely on these strategies to make progress on difficult problems. Thus if heuristics can be taught, they may help students become better problem solvers. Indeed, this hypothesis (among others) led Schoenfeld to develop the course.

Researchers in mathematics education have not found it easy to teach heuristics in the classroom (e.g., Lester, Garofalo, & Kroll, 1989). Schoenfeld has himself experienced difficulty at the college level. In his early work teaching problem solving, he identified three main complications in the task of teaching heuristics (Schoenfeld, 1985).

- *The specificity problem.* If heuristics are presented in their most general (and useful) form, students will be unable to apply them; if they are given in more context-specific forms, their number explodes and only some can be taught.
- *The implementation problem*. Applying heuristics requires many steps, and therefore creates many opportunities for students to make fatal errors.
- *The resource problem.* To be successful, students must know both the appropriate heuristics and the mathematics required to solve the problem.

In his continuing development and revision of the course, Schoenfeld has had to address each of these problems. A major goal of this paper is to understand his approach.

Effect

Schoenfeld has discussed the course many times (see [Schoenfeld 1983; 1985; 1988; 1989a; 1991; 1994] for the most substantive discussions). Some accounts have been descriptive introductions built around vignettes: rich snapshots of the class working on particular problems, such as the magic square (Schoenfeld, 1989a; 1991) or Pythagorean triples (Schoenfeld 1988; 1991; 1994). These accounts suggest that his students are engaged in more productive sorts of mathematical thinking and activity than are typical of most undergraduates.

Students work collaboratively in groups with or without Schoenfeld's presence—indicating engagement and commitment to the enterprise. They stop looking to him to evaluate the validity of their arguments, turning instead to their peers. They produce results that are new to them, surprising and interesting to Schoenfeld, and occasionally publishable (Schoenfeld, 1989b). And most important, they learn to use heuristics effectively over a range of problems, considering, pursuing, and monitoring multiple approaches.

Schoenfeld examined students' problem solving performance before and after the course using measures which ranged from paper-and-pencil tests to analyses of problem-solving protocols (see Chapters 7, 8, 9, and 10 of Schoenfeld, 1985). His results showed that students who completed the course (1) used a variety of heuristics effectively to solve challenging problems; (2) had a better sense of how to proceed and were less likely to "plunge in" with the first approach that came to mind; (3) saw through the surface features to the deeper mathematical structure of problems; and (4) used heuristics to solve problems unlike those they had worked previously in the course.

On the first day of class, Schoenfeld described one of the measures he used in his analysis:

> I gave an in-class final, and there were three parts to the final exam. The first part was problems like the problems we solved in class. No surprise, you expect people to do well on those. The second part was problems that could be solved by the methods that we used in class—but ones for which if you looked at them you couldn't recognize that they had obvious features similar to the ones that we'd studied in class. So yes, you had the tools and techniques, but you had to be pretty clever about recognizing that they were appropriate. And, the class did pretty well on those too. Part three of the final exam . . . There's a collection of books called the *Hungarian Problem Books* which have some of the nastiest mathematical problems known to man and woman. I went through those, and as soon as I found a problem I couldn't make any sense of, whatsoever, I put it on the final. (I know that makes you feel good.) [Laughter from class, Schoenfeld smiles.] The class did spectacularly well, and actually wound up solving some problems I didn't. . . .

Pedagogy

In the versions of the course we observed (1987 and 1990), the class was organized into six principal modes: lectures, reflective presentations, student presentations, small-group work, whole-class discussions, and individual work. In the first two class sessions all six modes occurred, although not exactly in the same proportions as throughout the semester. On the first day of the course Schoenfeld described these modes to the students:

> Most days I'm going to walk in . . . and hand out a bunch of problems. I've got enough here to probably keep us busy for two days or so. And what you're seeing here is unusual, because you won't be seated in rows watching me talk. Instead you're going to break into groups of three or four or five, and work on problems together. As you're working on them, I'll circulate through the room, occasionally make comments about the kinds of things you're doing, respond to questions from you. But, by and large, I'll just nudge you to keep working on the problems.

> Then at some point I'll call us to order as a group, and we'll start discussing the things that you've done, and talk about the things that you've pushed and why; what's been successful, what hasn't. I'll mention a variety of specific mathematical techniques as we go through the problems. Many of the problems are chosen so that they illustrate useful techniques. So you'll work on one for a while; may or may not make some progress; and then we'll talk about it. And as we talk about it what I'll do is indicate some of the problem solving strategies that I know, and that are in the literature, that might help you make progress on this problem, and progress on other problems.

Lectures. In contrast with most college classrooms, the lecture mode occurred relatively infrequently. When Schoenfeld lectured, the lecture segments were relatively short and oriented toward particular goals: to provide background on mathematical resources needed to make progress (e.g., mathematical induction), to introduce heuristics, and to describe his goals for the class. He did not generally present his own solutions to problems, except on the occasion that an important solution was not developed by the class. Because his lecture segments were short, pointed, and related to activities in other modes, many of the traditional effects of the lecture—e.g., student passivity and disengagement—were not evident. (More details are provided throughout the following sections by Meira, Smith and Kessel.)

Reflective presentations. In this mode, Schoenfeld presented mathematical commentaries to the class, interpreting segments of activity just completed and highlighting important aspects. We characterize them as “reflective” because they directed students’ attention to mathematically significant features of either Schoenfeld’s or his students’ actions. They differed from lectures because they engaged students as participants. They were unlike whole-class discussions because Schoenfeld pursued specific goals and directly controlled the flow. Reflective presentations took quite different forms: e.g. modeling a problem solution to illustrate a particular heuristic, to demonstrate a specific mathematical point, or to highlight executive control in problem solving; recounting, and highlighting aspects of students’ presentations of their solutions; conducting “post-mortem” reviews of complete problem solutions (see Schoenfeld, 1983 for a specific example). Like lectures, they all involved significant forms of teaching by telling;

i.e., substantive insertions of content into the classroom discourse (Ball & Chazan, 1994), but occurred in the broader context of problem solving. They provided students with a clear view of the reflective mathematical practices of a skilled mathematician, an opportunity that is absent from many college classrooms.

Student presentations. At appropriate junctures, students were invited to present their solutions to assigned problems. During these presentations, Schoenfeld avoided giving immediate verbal feedback and non-verbal evaluations of student success, though students initially expected such judgments (see Smith's analysis of the inscribed square problem). With a blank "poker-face" he usually addressed the class with one of the following questions: "What do you guys think?," "Does the class buy this argument?," or "Are you convinced?" These questions were routinely posed after each presentation to signal that students should not wait for an external authoritative judgment. Student presentations also provided opportunities to work on issues of mathematical exposition and communication; such as top-level descriptions of an argument vs. more polished and detailed versions, comparing formal/symbolic and informal presentations, contrasting convincing arguments with "hand-waving."

Small-group work. About 30% of each class was devoted to work in small groups of two to four students. Its purpose was to provide a stable and continuous context for students to engage collectively in problem solving. In the best of cases, this collaborative work generated negotiation among the members of the group about approaches to pursue, allowed each student to calibrate his/her own understanding of the mathematics involved with the other group members, and promoted the disposition to listen to and learn from peers. In this mode, Schoenfeld played the role of "traveling consultant" and critic.

Whole-class discussions. After an individual presentation or small-group work, Schoenfeld often engaged the class in collective discussion. Sometimes the class attempted to solve a problem as a whole group, and, as in the small-group work, Schoenfeld usually avoided immediate evaluation of the usefulness of the approach suggested by students, even when the approach could lead to a dead end. This mode had several purposes: it allowed all students to listen to each other's questions, comments, and solution attempts. As students started to feel more comfortable with the class, it slowly became a forum in which they could openly voice misunderstandings and/or requests for mathematical resources invoked by some and lacked by others. There were occasions later in the course in which the whole-class discussion also dealt with issues of mathematical elegance and aesthetics.

Individual work. Students had many opportunities to work individually before, within, and after some of the modes described above. However, individual work was the main mode for homework assignments, and the two take-home exams, on which students worked for about two weeks with the promise of not consulting each other. Individual work consisted not only in solving problems, it also included, as mentioned above, preparation for communication of results, either to a small group, the whole class, or (in the case of the written take-home exams) the teacher.

Students received specific guidelines about exams and grading. On the first day of class Schoenfeld told the students:

> [A] week or two into the class I'll give you the opportunity to write out a problem or two for me so that I can get a sense of the kind of writing you do, and give you some feedback on the kind of writing I expect. The first main thing we do is: about half-way through the course I'll give you a two-week take-home. It'll consist of about ten problems and they will occupy you for a long time. But you'll make progress on them and you'll do reasonably well on them. And then, the final. Again, the department formally requires me to give an in-class final, so I usually wind up giving a one-problem in-class final to meet the rules and regulations. That's about ten percent of the final exam grade. The rest of it is another take-home that you'll have two weeks to work on. There are some funny rules, which are that:
>
> What counts is not simply the answer, what counts is doing mathematics. And that means, among other things, if you can find two different ways to solve a problem, you'll get twice as much credit for it. If you can extend the problem and generalize it and make it your own, you'll get even more. The bottom line is, I'd like to have you doing some mathematics and I will do everything I can—including using grading—as a device for having you do that.

The 1990 class

Students

Mathematical Problem Solving, listed in the university catalog as Math 67, is not a required course for any major. The course prerequisite is one semester of calculus or consent of instructor.

The students in the first two classes had a wide range of mathematical backgrounds (see Table 1). For example, Jeff,[3] a history major, had taken one semester of calculus three years ago, and Diane, a genetics major, had taken the calculus sequence. In contrast, Mitch was a graduate student in computer science, and Jesús, a fourth year applied math major.

In the first week, the "traffic" in and out of the class was relatively heavy; students were shopping for classes and adjusting their schedules. The university catalog had also listed the course as beginning one hour later than it did, thus adding to the traffic. Thirteen students attended all or part of the first session. Three new students entered in the second session.

The eight students who completed the course were all enrolled for credit. Six were majors (or intended majors) in mathematics or computer science. Only one of these students (Jeff) had a major outside of science, mathematics, and engineering. Only one was female. The group comprised four European Americans, two Asian Americans, and two Hispanics.

Brief overviews of the first two class sessions

Session #1. For the first twenty minutes Schoenfeld introduced the course: its history, its basic mechanics, the grading system (a complete transcript is given in Appendix A). He then distributed the first set of problems and asked students to start working on them in groups. For the next twenty minutes most groups worked mainly on the first two problems: finding the sum of the telescoping series and inscribing a square in an arbitrary triangle. Forty minutes into the class, Schoenfeld called the class back and, for about twenty-five minutes, he discussed

[3]All students are referred to by pseudonyms.

Table 1

Background of Students Participating in Sessions 1 and 2

Student	Origin of Interest; Entry	Background
Austin	Saw description in the catalog Looking for such a course "for years" Entered at start of Session 2	Third-year computer science major Calculus sequence, discrete math Audited one class of Putnam[4] course
Devon	Saw description in the catalog Entered at start of Session 1	Third-year student with interests in math and computer science; part-time: Math 67 only course Older student, recent transfer to UCB
Don	Saw description in the catalog Entered at start of Session 1	Math major Calculus sequence plus 4 upper division courses
Jeff	Saw description in the catalog Entered at start of Session 1	Fourth-year history major; goal: teach history and math One semester of calculus as first-year student
Jesús	Saw description in the catalog Entered at start of Session 1	Fourth-year applied math major 13 math courses
Sasha	Saw description in the catalog Entered at start of Session 1	First-year student, intending computer science major Calculus, discrete math, Putnam course Math camps & 3 high school competitions
Stephen	Looking for some "easy units" Entered at start of Session 1	Physics major
Diane	Looking for a "fun class" Attended Sessions 1 and 2	Genetics major, calculus sequence Liked problem solving
Richard	Attended Sessions 1 and 2	Computer science major, part-time student Calculus sequence, 1.5 years before Worried about "rusty background"
Mitch	Read about the course Entered at start of Session 1, auditing	Graduate student in computer science Calculus sequence, discrete math, linear and abstract algebra
Sharon	Heard about Schoenfeld via family member Attended Session 1 only, auditing	Varied academic background; intending math major; Calculus, logic, and statistics;

the telescoping series problem. This discussion is analyzed in detail in Meira's section.

[4]The William Lowell Putnam Mathematical Competition is administered annually in December by the Mathematical Association of America to students who have not yet received a college degree (Reznick, 1994, p. 19). Neither of the students who attended the Putnam course (H90, Honors Undergraduate Seminar in Mathematical Problem Solving, also offered in the fall of 1990) mentioned taking the Putnam exam though they were interviewed early in 1991.

The students then went back to work in their small groups, and Schoenfeld moved around the class monitoring the work of each group. He discovered that some groups did not fully understand the written statement of the inscribed square problem, so he explained the distinction between showing that the required square exists and giving a construction. Approximately one hour and twenty-five minutes into the session, Schoenfeld again called the class back to a whole-class discussion of this problem which was interrupted when the class period ended. Students left class with instructions to think about the problem at home.

Session #2. Schoenfeld began by emphasizing the importance of working as a community and presenting results to one's peers. Upon request, one student volunteered to present his constructive proof that a square can be inscribed in an arbitrary triangle. His argument was fundamentally sound, but directed almost entirely to Schoenfeld who was standing at the side of the room. Schoenfeld noted this deference and explained that the class must become the judge of mathematical validity of proposed solutions. The student then addressed the class more directly, repeating his solution with slightly more detail. This discussion lasted twenty-five minutes. Smith analyzes in detail the way in which Schoenfeld directed the discussion on this problem.

Schoenfeld then drew students' attention to the third problem of the set, placing the integers from 1 to 9 to make a 3 by 3 magic square. Another student volunteered to present his solution, and its correctness was immediately apparent. The next forty minutes were spent in discussion of different solution paths that could produce the same result. Schoenfeld led this discussion, involving students in substantive ways and introducing many new heuristics. Kessel analyzes this discussion in detail.

The final fifteen minutes of the session were devoted to quick solutions of the next problems in the set, which will not be discussed in this paper. In sum, the next three sections of this paper cover most of the whole-class instructional episodes of the first two two-hour class periods.

Presenting and doing mathematics: An introduction to heuristics

Luciano Meira

Solving problems is a considerable part of what mathematicians do, and learning to solve problems is part of learning to think mathematically. Shaping the culture of the classroom so that his students learn to think mathematically is the heart of Schoenfeld's teaching enterprise. Therefore his central goal is to create a classroom community which embodies selected values, beliefs, and activities of the professional mathematical community.

But within this parallel, Schoenfeld has also acknowledged the individual and collective differences between professional mathematicians and the students taking his course.

> The class itself is a mathematical community (better, a micro-community in which certain mathematical values are highly prized) in which the students interact with each other in ways very much like the ways that mathematicians interact—but at a level appropriate to their knowledge and abilities. At their own level the students *are* mathematicians, engaged in the practice of mathematical sense-making. They *do* mathematics, with the same sense of engagement and involvement. The difference is that boundaries of understanding that they challenge are the boundaries of their own (community's) understanding, rather than those of the mathematical community at large. (Schoenfeld, 1988, author's emphasis)

His characterization of the relevant differences centers on issues of mathematical background. Students' views of problems and significant results reflect their own understanding, which is substantially more limited than professional mathematicians'. But are the differences between these two communities only a matter of what constitutes shared knowledge and problems at the edge of collective understanding? We think not, especially in the first sessions of the class when much "shaping" of the community is being done. We propose that there are other important (and sometimes subtle) differences that follow directly from a second and equally obvious difference, that the classroom community is deliberately shaped and "engineered" by Schoenfeld, whereas the professional mathematical community has no recognized single authority or leader.

To explore the subtleties of these differences, we introduce the distinction between "presenting" and "doing" mathematics. "Doing mathematics" means engaging in reasoning that reflects the thinking of mathematicians: resourcefully tackling and making progress on hard mathematical problems. In order to bring students to the point where they can approximate mathematical doing, some "presenting" must take place, in both traditional and less traditional forms. As will become evident below, we use "presenting" to characterize different acts of teaching, though all forms involve the display of some mathematical concept or part of mathematical practice for students.

The contrast between doing and presenting mathematics is enacted by Schoenfeld during work on the first problem, summing the telescoping series. We analyze this contrast in three consecutive segments of class activity: (1) his mock lecture on the standard solution to the problem, where Schoenfeld critiques a traditional form of teacher presentation in college mathematics classrooms; (2) his presentation of a heuristic-based solution as an important part of the practice (the "doing") of professional mathematicians; and (3) his lecture on and subsequent use of mathematical induction to prove that the solution found is general.

Caricaturing mathematics teaching as presenting: The mock lecture

Finding the sum of the telescoping series is a well-known problem, appearing in most first-year calculus courses. It asks for the sum of the following terms,

$$\frac{1}{1\ x\ 2} + \frac{1}{2\ x\ 3} + \frac{1}{3\ x\ 4} + \frac{1}{4\ x\ 5} + \ . \ . \ . \ + \frac{1}{n(n+1)}.$$

In placing this problem first, Schoenfeld's apparent goal was to demonstrate the value of heuristics as tools to unpack results which are either unknown or recalled but not understood.

After some twenty minutes of group work, where students worked on the telescoping series and other problems in the set, Schoenfeld called the class together to present a "lecture" on the textbook solution. This was not just any lecture, but a play that caricatured the "typical" calculus professor presenting the standard solution to the problem. We quote the transcript at length so that the content and tone of the play are clear. Note that the goal of differentiating between presenting and doing mathematics was explicit from the start.

Let me show you what you were shown, by metamorphosing into the typical calculus professor for three minutes, and lecturing on the solution to that problem as it's typically presented in a calculus class and then talk about the way it gets *done* rather than the way that the solution actually gets *presented*. Well, I won't quite be the typical mathematics professor, 'cause I won't mumble at the board [giggles from class] but . . . it goes something like this. [Schoenfeld walks toward the door, turns, and starts back toward the board as if he were another person.]

All right. Well, the problem I asked you to look at was find the sum: [writes rapidly, banging the chalk, on the board and states the formula simultaneously]

$$\sum_{i=1}^{n} \frac{1}{(i)\ x\ (i+1)} = \frac{1}{1\ x\ 2} + \frac{1}{2\ x\ 3} + \frac{1}{3\ x\ 4} + \ . \ . \ . \ + \frac{1}{n(n+1)}.$$

Now, all you need to know is the obvious algebraic observation that [he writes and speaks simultaneously]

$$\forall i \ \frac{1}{i\ x\ (i+1)} = \frac{1}{i} - \frac{1}{i+1}.$$

That's trivial. You can check it algebraically, OK; I don't waste my time with such things at the board.

[Writes as he speaks; the italicized text below is what also is written.]

Now that says that 1 over 1 times 2 [he points to $\frac{1}{1\ x\ 2}$ in the first formula]

is equal to $\frac{\mathit{1}}{\mathit{1}} - \frac{\mathit{1}}{\mathit{2}}$,

[A student, perhaps recognizing this, says, "Ah . . . yeah."]

$$\frac{1}{2\ x\ 3} \text{ is } \frac{\mathit{1}}{\mathit{2}} - \frac{\mathit{1}}{\mathit{3}},$$

$$\frac{1}{3\ x\ 4} \text{ is } \frac{\mathit{1}}{\mathit{3}} - \frac{\mathit{1}}{\mathit{4}}.$$

[He writes "+ . . . +".]

The next to the last term is $\frac{1}{n-1}-\frac{1}{n}$ and the last term is $\frac{1}{n}-\frac{1}{n+1}$.

[The following formula (*) is now on the board:]

$$(\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\ \ldots\ +(\frac{1}{n-1}-\frac{1}{n})+(\frac{1}{n}-\frac{1}{n+1}).$$

Got that? [no pause] All right [giggles from class].

Well, now all we need to do is make the observation that [he pushes board[5] upon which he has been writing under top board so all but (*) is covered] I've got minus a half and plus a half next to each other, and they cancel [he strikes through canceled terms], I've got minus a third and plus a third next to each other and they cancel, minus a fourth and plus a fourth and they cancel, this [pointing to the term one *n*th] cancels with the previous term—minus an *n*th plus an *n*th and they cancel—so the only terms left are the first one, and the last one, namely 1 over $n + 1$. [Formula (*) now looks like:]

$$(\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\ \ldots\ +(\frac{1}{n-1}-\frac{1}{n})+(\frac{1}{n}-\frac{1}{n+1}).$$

[He pushes the board upon which he has been writing up so it is completely covered by the board above.] The first one was 1 over 1, minus the last one: 1 over $n + 1$; or . . . n over $n + 1$ . . . [writes formula (†)]

$$= 1-\frac{1}{n+1}=\frac{n}{n+1}\ \text{Q. E. D.}$$

Q. E. D. You all know what Q. E. D. stands for, don't you?

Student: *Quod Erat Demonstrandum*.

Yeah, the Latin is *Quod Erat Demonstrandum*, "that which was to be shown"; the English . . . "Quite Easily Done." OK.

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[5]The classroom has a blackboard with three sections. The central section has two panels which slide vertically over a third fixed panel. Schoenfeld writes the first three formulas on the lower of the two sliding panels, and the remaining formula (†) on the fixed panel.

> Now, *I* can do things like that, that's why *I'm* a mathematician. We don't expect *you* to do things like that but *you* can memorize them. That's why *I'm* up in front at the board and you're down there. OK.
>
> End of lecture, see you next time. [He pauses; returns to normal demeanor.]

We take the overall purpose of the play to be straightforward. In acting out an objectionable teaching practice, Schoenfeld sets the stage for presenting himself and his course as a new and more positive mathematical experience for students. We identify the following negative elements in this caricature of "traditional" teaching:

- The caricatured professor stated the problem exactly as it is printed in the problem set and began his lecture without any preliminary discussion of the problem that might engage students in the task. In traditional mathematics classrooms, the curriculum (problems and solutions) is seen as uniquely defining the activity.
- He spoke and wrote on the board very rapidly. Such high speed deliveries have additional inhibitory effects on student contributions, over and above the standard expectations, among both faculty and students, implying that interruptions to lectures should be minimized.
- He assumed that the key algebraic reformulation was obvious and never addressed the source of the insight, suggesting only that students could undertake the remedial task of "checking" its validity.
- He maintained a haughty and arrogant attitude throughout, but especially in his implication that his students had seen the solution in calculus but apparently forgotten it and in his emphasized difference between professors who know and students who memorize.
- He asked no real questions during the lecture. The two queries posed to the class were not serious invitations to discussion, since he did not wait for a student response. These "questions" were merely rhetorical ornaments in the lecture.
- It was evident from the videotape record that he wrote the key algebraic steps on panels of the moveable chalkboard but then quickly removed them from the students' view behind the fresh panels he was sliding into

> place. On one occasion, he covered a long computation just as he began to summarize it.

The mock lecture was clearly a set-up, a worst-case scenario of traditional mathematics teaching practices where teachers "tell" students the facts and procedures they think are important and students memorize them. Schoenfeld's enactment of this caricature communicates at least two related messages to his class: "I know about your experience with the mathematics teachers, particularly college professors" and "I will make your experience with me different." As is evident from the start, the mock lecture serves as counterpoint between students' impoverished past experience and the yet-to-be seen, but allegedly real mathematical practice. What properties of real mathematical practice are effectively modeled by Schoenfeld in these first sessions of his class? How does he say that mathematics "gets done"?

Presenting professional mathematical doing

Immediately after the mock lecture, Schoenfeld presented a heuristic-based solution that "serves as a window into the practice of mathematicians." He began by declaring that mathematicians do not solve problems by recalling algebraic identities, but by applying well-known problem solving strategies:

> Now, if I called up any member of the math department at four in the morning and said, "Hey, your house is on fire but before you leave, what's the sum of this series?" they could tell me because it's part of the mathematicians' collective unconscious. If I gave them a slightly more complicated problem, being mathematicians, they'd probably stop to solve it before they ran out of their house anyway—they're a little weird that way—and what they'd do is *not* pull a rabbit out of their hat, the way I just did to solve this one, but rather to use a reasonably, (well, a very) well-known and quite comfortable strategy to all of them, which is [begins writing] long-winded, but it's worth writing down.
> [Writes as he speaks; the italicized text below is what also is written.]
> *If you've got a problem you need to make sense of and it has an*—jargon coming up!—*integer parameter n,* that is something that takes on values . . . (whole number values, and I'll be explicit about what that is in a minute) . . . *try values of n = 1, 2, 3, 4, 5, . . . and see if you can find a pattern.* That pattern

may suggest what the answer is and may even suggest how you can verify it—if that's the answer.

This is the strategy. It's our first official strategy of the course (put it in a box to make it pretty) [he draws a box around what he's just written] and introduces the first serious piece of jargon for the course. It's called a *heuristic strategy*.

[The following is on the board:]

> If you've got a problem you need to make sense of and it has an integer parameter n, try values of $n = 1, 2, 3, 4, 5, \ldots$ and see if you can find a pattern.

Heuristic strategy

In our analyses, we will refer to this heuristic as "if a problem has an integer parameter, try specific values and look for a pattern," or Try Specific Values for short.

Schoenfeld was quite explicit about how Try Specific Values could be applied to this problem. "This [problem] asks for the sum of n terms, and you can ask: What's the sum of the first one, the first two, the first three etc." He wrote the first four partial sums and retrieved their results from the class,

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$$

and suggested that the next sum would be $5/6$. With the emergent pattern in hand, he queried the class about verification.

AS: So at this point, I've seen the pattern, convinced myself on the basis of a number of examples that it's probably right. Am I done?

Sasha: Can you prove your result?

AS: Probably. [He pauses, someone, perhaps Sasha, laughs.] Patterns *can* be deceiving, I mean this is pretty compelling evidence. But we'll see some examples later this semester where they're not, where a compelling pattern doesn't necessarily come true. Since this is a math class, I feel a moral obligation to actually confirm that the pattern holds. How do I do that?

Sasha: Math induction.

Two features of this segment are worth comment. Schoenfeld's query to the class, "Am I done?" was his first real question, a serious request for students' contributions to the solution.[6] He invited students to collaborate, modeled a standard move in mathematical thinking where solved problems are beginnings, not ends, and drew a sharp contrast with the haughty, one-sided nature of his mock lecture. This question signaled that his own teaching, not the traditional calculus instructor's, was beginning. But with this overture to students, he also explicitly indicated that he had sure command of the course content (some patterns would deceive) and a ready proof of the pattern of partial sums. In short, he presented the use of heuristics, a key component of real mathematical activity, and began to engage the students as thinkers in the real practice of mathematics, without straying off the path to a complete solution.

Presenting some mathematics

Following Sasha's suggestion, Schoenfeld asked:

> How many of you guys feel comfortable with induction? [pause, students are not visible or audible on the videotape at this point] OK, let me ask it the other way: How many of you feel uncomfortable with induction? [pause] OK, good. [erases blackboard panel, pushes it up] I'm not going to spend too much time on it in the course but it will occasionally be a useful tool. So I'll

[6] Schoenfeld's requests to students to calculate the partial sums, $1/2 + 1/6$ and $1/2 + 1/6 + 1/12$, were not questions in the same sense. Because students' knowledge of the addition algorithm for fractions could be assumed, only the speed of their response—not the content—was an issue.

give you an example of how it works here and if it turns out to be an issue, then come talk to me in my office hours and we'll worry about it then. OK?

He presented the principle of mathematical induction and used it to show that the emergent pattern in the partial sums holds for all positive integers n. This exposition was straightforward, except for two important features: his use of a staircase metaphor to help the students make intuitive sense of induction and his increasing requests for students' contributions to the evolving proof by induction.

The general idea of mathematical induction was stated in the standard manner but also characterized as "the mystical algebraic formulation." "If you'd like to show something is true for, in the simplest case, all whole numbers, 1, 2, 3, 4. . . . prove the statement is true for $n = 1$ [first inductive hypothesis]; and prove that if it's true for $n = k$, then it must also be true for $n = k + 1$ [second inductive hypothesis]."

To illustrate induction, he drew a staircase on the board and said that, if one were on any given step of the staircase, the second inductive hypothesis allowed jumping to the next step, while the first inductive hypothesis allowed one to get on the staircase.

> This [second] part, if it's true for $n = k$, then it's true for $n = k + 1$, where k can be anything, is a funny inductive assumption . . . so that's an assumption that allows you to jump from the third floor to the fourth floor. . . . Now in and of itself, that statement may or may not do you any good. It simply says, if you've managed to show that it's true for some value, then it's true for the next value as well. . . . The problem is getting on the staircase in the first place. . . . That's why you need the first part. . . . You show that if the statement is true for $n = 1$, I can get on the staircase. . . . So, that's the two parts to an inductive argument: first there's the place where you get on the staircase; second, you can climb one step at a time.

The sum of the telescoping series was then solved for the third time. Schoenfeld rewrote the problem statement, proved it for the trivial $n = 1$ case, assumed the statement true for $n = k$, and proved it for $n = k + 1$ by adding the $k + 1$st term ($k + 1$) to both sides of the $n = k$ equation and simplifying the resulting algebra. He requested student contributions many times in building the inductive assumption:[7]

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[7]These student(s) could not be identified from the videotape.

AS: Then what do I want to show?

A student: [Inaudible] prove it's true for $k + 1$.

AS: OK, so I assume this is true, what I want to show is: And now the question is: How do I write this statement for n equals $k + 1$? [No response.]

AS: What does the left-hand side of it look like?

A student: The same as [inaudible] but with one more term.

AS: Yeah, it's going to look the same except the last term is going to be $k + 1$, 'cause I'm doing it for $n = k + 1$, $(k + 1)(k + 2)$. OK. So what I want to show is: 1 over 1 times 2, plus—and I'll write the next to last term—to make life easy for myself. The last term is 1 over $k + 1$ [times] $k + 2$. The next to last term is going to be what? . . . What's the right-hand side going to be if n is equal to $k + 1$?

A student: $k + 1$ over . . .

The solution of the telescoping series came to an end with Schoenfeld's transition back to the more general application context for Try Specific Values:

> When you see an n, sometimes it'll be explicit as it is here, sometimes it'll be implicit, you just look at it and say to yourself, "Hey, it's *really* a problem that has different values for n equals 1, 2, 3, 4, 5, even though there's no n in the problem formulation." Then, if you need to make progress on it, it often helps to look for systematic patterns.

Discussion

Our characterization of the telescoping series problem has interwoven two levels of analysis of Schoenfeld's teaching. First, we have emphasized the many introductions he makes in this first session: to his own teaching as different from other, content-oriented, classes; to heuristic strategies as a crucial component of problem solving; and to his mastery and expertise both as a teacher and as a mathematician. But none of these features is surprising, since they are described in Schoenfeld's own written accounts of the course (1983; 1985; 1991; 1994).

We believe our contribution consists in providing another level of analysis. We have used the distinction between "doing" and "presenting" to capture the more fundamental connection between his teaching and the practice of mathematicians that underlies those introductions. Figure 1 presents a model of Schoenfeld's teaching with the telescoping series as relationships between doing and presenting mathematics.

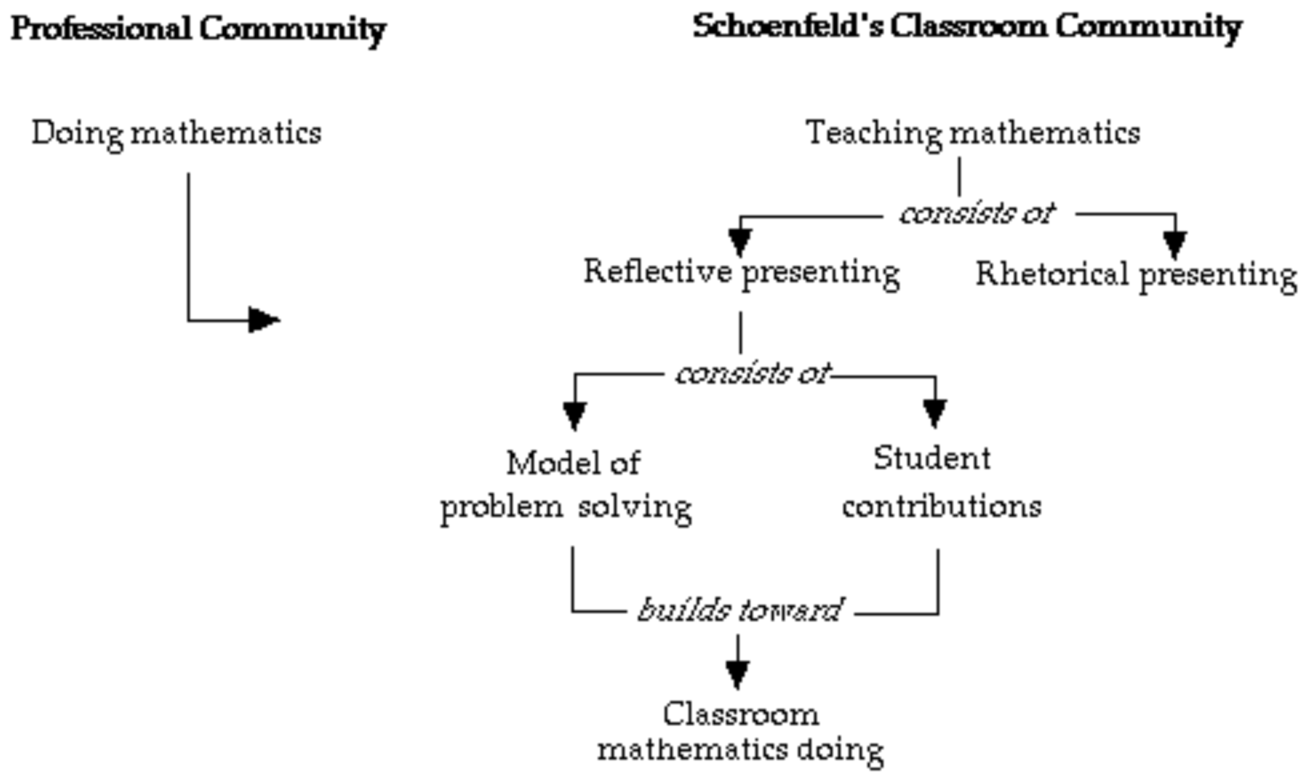


Figure 1
Schoenfeld's pedagogy as doing and presenting mathematics

At the top level, the model distinguishes the practice of mathematicians (professional "doing") from teaching. Despite Schoenfeld's intentions to construct a classroom community to parallel professional mathematical practice, he was a "messenger" who modeled professional practice for his students, probably because he thinks that at the beginning, modeling is a way to help that community to emerge. In contrast to that professional "doing," Schoenfeld's teaching in this segment of the class involved two forms of "presenting": *reflective* and *rhetorical*. We characterize his mock lecture as a rhetorical presentation because it was a skillful performance that he enacted alone to encourage students to follow his lead in developing contrasting forms of classroom dialogue and practice. His reflective presentations, as exemplified in his discussions of Try Specific Values and mathematical induction, were more frequent and characteristic of his teaching throughout the course. They involved his modeling of significant parts of skilled problem solving including both knowledge and decision-making and his solicitation of student contributions within these goal-directed activities.

We consider them as “reflective” for different reasons. Some presentations introduced new knowledge (e.g., Try Specific Values) in particular problem contexts and highlighted for students the importance of thinking when and where that knowledge can be used again (recall Schoenfeld’s closing comments on Try Specific Values). The reflection here involved the relationship between knowledge and the (problem) contexts where that knowledge is applied. In other cases, he modeled specific actions of a skilled problem solver, e.g., control questions such as “Am I done?” Reflection there involved an awareness of the relation between the state of a person’s evolving solution and the problem context. All such presentations contained (1) an introduction to a necessary “tool of the trade” in one context, and (2) pointers to how knowledge and skill relate to a wider range of contexts. The intent of these presentations was to help students carry away from the course a different form of classroom mathematical practice.

Because they were the places where Schoenfeld presents new knowledge and skill to students, it is important to contrast reflective presentations with simpler forms of “transmission” teaching common to college and precollege instruction in mathematics. Both involve teachers’ display of mathematics knowledge and skill for students who are understood to lack that knowledge and need it to make further progress. We see elements of transmission in the introduction and use of Try Specific Values and in the review of mathematical induction. Moreover, the context and content of Schoenfeld’s reflective presentations differed from traditional “teaching by telling” as well as “socratic teaching” in significant ways. In emphasizing process in problem solving, he shifted the focus from mathematical *content* to issues of how mathematics is *done.* His solicitation of student contributions was a step toward fulfilling the expectation, stated during the first twenty minutes of the course, that students would soon take over and use the tools he presented without his assistance. In our view, it is the complex interweaving of transmission and student participation in Schoenfeld’s reflective teaching combined with his explicit statements and illustrations of the goals of his actions that makes his course a good example of teaching towards sense-making.

Finally, it is worth emphasizing that our analysis has not centered on the practice of the mathematical community (professional “doing”), but on “presenting” in the classroom context. Indeed schools are unique and specialized contexts for mathematical thinking. We think the analysis shows that the differences between his classroom community and that of professional mathematicians lies not only in shared knowledge and skills, as Schoenfeld has suggested, but also in the *nature and goals* of the practices in these two contexts. His presentations of mathematical content and heuristics had clear instructional goals and employed rather tightly controlled mechanisms for student participation. His intent was that students construct appropriate models of and beliefs about mathematical activity in the professional community. But this teaching practice did not make his classroom part of the professional mathematical community. Rather it helped to close the gap between the two, creating in students the sense of belonging and contributing to a more authentic mathematical practice.

Making the case for heuristics:

Authority and direction in the inscribed square

John P. Smith III

To teach his students to solve challenging problems, Schoenfeld must himself solve a difficult instructional problem: how to introduce Pólya-type heuristics so that students quickly appreciate their power and slowly learn to apply them productively across a wide range of problems.

As his own past teaching and research has shown, this problem does not submit to easy solutions (Schoenfeld, 1985; 1992a). Students can struggle to see how and where to apply particular heuristic strategies because of their general character as "rules of thumb." Schoenfeld could address this part of the problem by presenting more specific versions of each strategy with clearer conditions of application. But if he did, the list of useful heuristics would become too long and cumbersome to teach and learn (the "specificity" problem). So the generality of strategies (and their attendant vagueness) must be retained. Given the generality of these strategies, students must be thoughtful in selecting, applying, and evaluating them, but such thoughtfulness is difficult to teach. How, for example, should students evaluate their work so that they avoid committing too much time to unproductive approaches? Even if students select productive heuristics, applying those strategies usually involves many steps, and mistakes at any one point can undermine the entire effort (the "implementation" problem). Finally, the skillful use of heuristics is not neutral with respect to content knowledge. If students do not know or cannot recall the necessary mathematical concepts or procedures, even workable solution plans can fail (the "resource" problem).

Many curricular approaches to problem solving fail to take these problems seriously. It is common, especially at the pre-college level, either to cast problem solving as recreation—a separate activity from the "real" task of learning procedures (e.g., Cooney, 1985)—or to teach students to practice and master each strategy separately. These efforts, as Schoenfeld (1992a) has argued, fundamentally miss the mark.

> Problem solving in the spirit of Pólya is learning to grapple with new and unfamiliar tasks when the relevant solution methods (even if only partially

> mastered) are not known. When students are drilled in solution procedures . . . , they are not developing the broad set of skills Pólya and other mathematicians who cherish mathematical thinking have in mind. (p. 354)

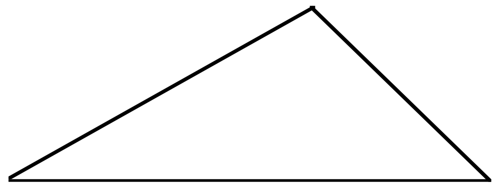
One central element in teaching problem solving is identified here: Students must regularly work on real problems, not "exercises" that are clearly tied to standard procedures or methods. But even with such problems in hand, how then can you teach problem solving, introduce and highlight heuristics as important "content," and avoid the pitfalls identified above?

In this section we analyze one important step Schoenfeld made toward solving this problem, using the solution and discussion of the challenging inscribed square problem as data. Our main claim is that Schoenfeld's approach involved leading the class through the solution in a carefully planned and directive manner. In so doing, he acted in accord with traditional classroom norms (e.g., the teacher is the mathematical authority) that he aimed to undermine and change. Though his students eventually chose their own approaches to problems and evaluated their attempts and solutions (and those of their peers), they were shown their way through these issues on this particular problem. Schoenfeld's choice to play the strong leader and director indicates that his actions and local goals early in the course did not map onto his long-term intentions and achievements in any simple way. Getting the problem solving class "off the ground" was a quite different task than teaching it in its mature, stable form—the stage emphasized in his written accounts (Schoenfeld, 1988; 1989; 1991; 1994).

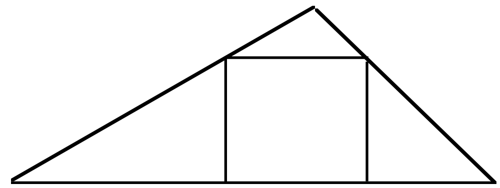
The problem and two relevant heuristics

The task of inscribing a square in an arbitrary triangle was second on the problem sheet after the telescoping series. Its wording and accompanying diagrams are reproduced below (Figure 2).

> You are given the triangle on the left in the figure below. A friend of mine claims that he can inscribe a square in the triangle—that is, that he can find a construction that results in a square, all four of whose corners lie on the sides of the triangle. Is there such a construction—or might it be impossible? Do you know for certain that there's an inscribed square? Do you know for certain there's a construction that will produce it?



The given triangle



What you'd like to get

Is there anything special about the triangle you were given? That is, suppose you did find a construction. Will it work for all triangles, or only some?

Figure 2
Statement of Problem 2:
Inscribing a Square in an Arbitrary Triangle

The questions posed in the problem statement raise the issues of existence and construction. First, there is the problem of showing that a square can be inscribed in the given triangle. But an existence proof may not necessarily lead to the construction of the square, so the question of whether the inscribed square can be constructed using Euclidean ruler and compass techniques remains. The existence/construction distinction influenced the class's work in two ways. Students struggled at first to understand the problem statement, and a major part of their confusion was their difficulty in separating these two issues. Schoenfeld also used the distinction to structure the discussion. He drew on his knowledge that some existence arguments generate constructions more easily than others to support the students' progress toward a solution (see Schoenfeld, 1985, pp. 84–91 for his analysis of the problem).

In contrast to the preceding problem (summing the telescoping series) and the subsequent one (the 3 by 3 magic square), the class found both parts of the inscribed square challenging. Much of the students' work prior to specific suggestions from their teacher was devoted simply to understanding the problem. The difficulty they experienced in getting started provided Schoenfeld with an early context for demonstrating the power of heuristics in solving problems.

Two related heuristics, both attributed to Pólya, were introduced in solving this problem. The very general strategy, *Look for a related problem that is easier to solve and try to exploit its solution to solve the original problem*, was the first. Schoenfeld's presentation of it was cautionary, if not somewhat negative. A number of "easier, related" problems were generated and found to be either hard to solve or difficult to exploit to solve the original problem. These cautions set up the second heuristic, *If there is a special condition in the problem, relax that condition and look for the desired solution in the resulting family of solutions.* Two "special conditions" are embedded in this problem: (1) that the inscribed figure must be a square (a rectangle is easier) and (2) that it must have all four vertices on the triangle (three vertices are easier). Relaxation of either condition can (and did) produce an existence proof.

The classroom solution

Overview. Schoenfeld's work with this problem can be divided into four phases, spanning some 70 minutes of class time. Some students worked on the problem during the first 20 minutes of group work prior to the discussion of the telescoping series solution. Immediately following that discussion, Schoenfeld directed the class, again in groups, toward the inscribed square problem with the strong "hint" to Solve An Easier Related Problem (Phase 1). When he called the groups back together to discuss their progress (Phase 2), three different related problems were suggested, two by students, one by Schoenfeld. They were considered and ultimately rejected. The second heuristic, Relax a Condition, was then introduced and applied in Phase 3. The two conditions were identified, and Schoenfeld used each to produce an existence proof. At the end of the first session, he sent the class away with some general directions to investigate one of the existence proofs more closely. The second class session opened with Devon's constructive proof (Phase 4). Schoenfeld used his solution as context to state some features of the mathematical community he desired.

Clarifying the problem and presenting Solve an Easier Related Problem (Phase 1). When the discussion of the telescoping series problem ended, Schoenfeld directed the class to the inscribed square.

> This [telescoping series] is a fairly straightforward example. We'll encounter a lot more later in the class that are not so straightforward. What I want to do to nudge you in the direction of a solution to problem 2, but not get you far enough yet most likely, is mention a second strategy and have you think about problem 2 a little bit. Which is: [writes and speaks] *If you can't solve the given problem, try to solve an easier related problem and then exploit your solution.* That's a statement that's almost verbatim what it comes out of—from Pólya. See if you can use that to solve problem 2.

The arrangement of the class at this point—still in small groups—is reproduced below (Figure 3).

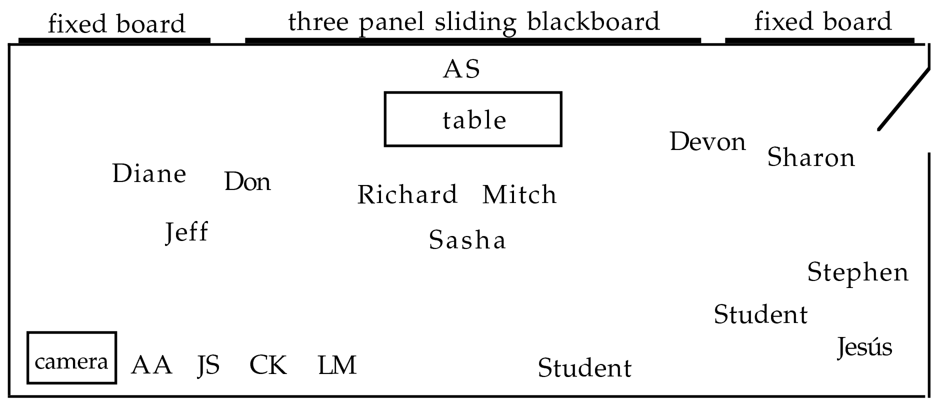


Figure 3
Arrangement of the Class in Small Groups
Session 1: Inscribed Square

The class did not immediately move to this task. Snippets of conversation from the four groups indicate that each returned to discuss the telescoping series solution, solve the series problem that was given just after the telescoping series problem,[8] or discuss the principle of mathematical induction. About halfway into these 20 minutes, one group after another turned to the inscribed square problem. Schoenfeld worked his way around the room, discussing the problem statement with each of the four groups. Members of two groups questioned him about what exactly the problem was asking. His responses emphasized the distinction between existence and construction. These quick "check-ins" not only assisted students in understanding the problem but allowed Schoenfeld to observe the work of each group and see which particular "easier related" problems they generated.

In opening the whole group discussion, Schoenfeld declared his expectation that students would come to the board in the next class session and present their work on problems. He then focused their attention on the inscribed square problem with a question, "What does problem 2 tell you to do?" When there was no immediate response from the class, he explained the two parts of the problem and used the example of an angle bisector to distinguish existence from construction. The existence of the angle bisector was established by considering all rays interior

[8]The second part of problem 1 was, "For those of you who've seen this series, how about $1/2! + 2/3! + 3/4! + 4/5! + \ldots + n/(n+1)!$?"

to the angle whose endpoints are the vertex and using a continuity argument. He then drew students' attention back to Solve an Easier Related Problem, reminded them that they would be reading Pólya's discussion of it in *How to Solve It* (a supplemental reading for the course). But his stance was cautionary, "... remember I said lots of people thought Pólya didn't quite work and this is an example of why. We need to push him a little bit further." Even as he promoted heuristics, he hinted at difficulties in using them, in this case the specificity problem.

Evaluating Solve an Easier Related Problem (phase 2). When Schoenfeld asked a more specific question, "What easier related problems did people try? I'm curious," three students responded. For each suggestion, Schoenfeld gave a quick verbal restatement and wrote the suggested approach on the left-hand blackboard (leaving the center board empty and available). We reproduce the major elements of this dialogue below because it is crucial to understanding and interpreting his approach.

Mitch: Relax the constraint on the square and try a rectangle.

AS: OK, let me get to that in a minute [chuckles]. OK, um. So the problem says, stick a square inside the triangle and each corner is on the triangle, one easier related problem is: Don't go for the square, go for a rectangle. [Writes "1. Try a rectangle instead" under the list heading "Related Problems."]
What else did people try? I saw people doing different things, so I know that you tried.

Devon: I tried to disprove it for an arbitrary triangle.

AS: OK. Try to find a counterexample. So, this is if you don't believe it's true, that's not part of the strategy but, [Writes as he speaks, "Look for a specific counterexample," well below suggestion #1] And that's a generally useful thing to do. [Some of AS's short commentary on the benefits of looking for specific examples and counterexamples is omitted here.] Other things that people tried? Yup? [responding to student]

Student: A circle. [This student was not visible on the videotape.]

AS: OK. The problem was, stick in a square inside a triangle. An easier related problem might be, [writes as he speaks, "Try a circle," just below suggestion #1]. Other things that people tried? Yeah? [responding to Sasha]

Sasha: Is that a square in a circle or a circle in a triangle?
[AS appeals to the student who made the suggestion and clarifies that suggestion #2 was to inscribe a circle in a triangle. He outlines the construction of the square in the circle but states it is not useful for solving the problem.]

AS: Other things people might have tried?
[no response for approximately 5 seconds]
I can mention at least one more that I thought I saw people doing, and that I've certainly seen before. Instead of making an arbitrary triangle, make a special kind of triangle, try either the isosceles or equilateral triangles. [He writes, “3. Instead of an arbitrary triangle, try special triangles—isosceles, equilateral.”] Let me leave number 1 alone for a short while. I'll get back to that and a couple of others, and talk about the general process and illustrate it with numbers 2 and 3. ‘Cause this is a general discussion of, what happens when you try to use the suggestions.

The final written list of suggested approaches to problem 2 is reproduced in Figure 4 below.

<u>Related Problems</u>
1. Try a rectangle instead.
2. Try a circle in a triangle.
3. Instead of an arbitrary triangle, try special triangles—isosceles, equilateral.

Look for a specific counterexample.

Figure 4
Schoenfeld’s Restatements of Students’
Easier Related Problems for the Inscribed Square

Two important teaching decisions are notable in this exchange; both are related to the task of structuring the discussion of Solve an Easier Related Problem and the problems it generated. First, Schoenfeld did not list the suggestions in the order in which they were given. The suggestion to look for a counterexample was listed below the others and separated from them by a squiggly line. The message seemed to be: this is a different sort of suggestion, and we should treat it differently. Second, he chose not to consider the suggestions in the order in which they were given. At the end of the interchange, he declared his intention to "discuss" suggestions 2 and 3 before suggestion 1, though the latter was certainly a straightforward example of an "easier related" problem. These choices suggest that Schoenfeld had lessons he wanted to draw from this problem, that some student suggestions fit more easily with his plan than others, and that to draw out these lessons he needed to consider the students' suggestions in a particular order.

To frame the discussion of the "general process" of using related problems (and the attendant pitfalls) he drew an application scheme for the second heuristic on the board and identified a question relevant to each step. For the first step, if you can identify what seems to be an easier related problem, can you solve it? For the second and perhaps more important step, does that solution help to solve the original problem? This scheme is reproduced in Figure 5 as he drew it on the board.

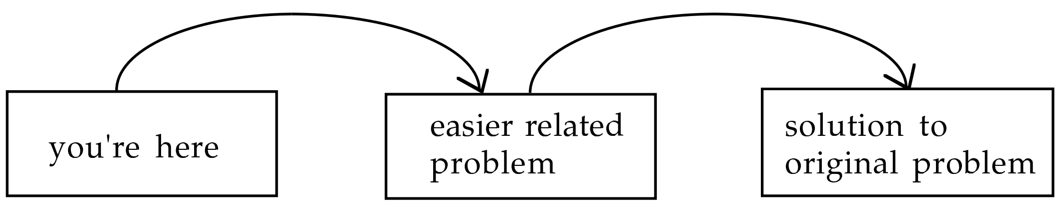


Figure 5
Schoenfeld's Application Scheme
for Solve an Easier Related Problem

With this general frame before the class, he sketched a circle inscribed in a triangle, stated that the construction could be done (referring again to work in high school geometry), but declared that it could not be used to solve the original problem.

> So for that particular problem, this part is easy [tracing the arrow between the "you're here" and the "easier related problem" box] at least at the level of yes, you can take that step. . . . but *I've never found anyone* who was actually able to take that particular thing, go from having a circle inscribed in a triangle and be able to use that to inscribe a square in a triangle. So the problem is that you can spend a fair amount of effort getting here [pointing to the "easier related problem" box] and then *I know of no way to get there*

> [pointing to the "solution to original problem" box]. So that's an example of a stepping stone that only doesn't do you too much good because it only gets you halfway there. [emphasis added]

His treatment of the two special triangles which followed was similarly brief. Instead of drawing either an isosceles or equilateral triangle on the board, he simply stated that both possibilities fail by both criteria, easier and related.

> It looks like it should be easier to inscribe a, square in something nice and regular like an isosceles triangle instead of a random triangle, or maybe even equilateral, but it turns out *I don't know of anyone who has actually managed to do that in an easy way, and I don't know of anyone who's been able to show how you can go from a solution of that to the general solution*. [emphasis added]

From these illustrations of the problematic nature of Solve an Easier Related Problem, Schoenfeld stated his general point, "When you're working on a complicated problem that involves using a stepping stone, you want to think both about getting to the stepping stone and whether or not you can get from there on." The stage was now set for reformulating this heuristic in a more specific and deterministic form.

Presenting Relax a Condition (phase 3). Declaring his intention to help students be more specific about how they might generate easier related problems, Schoenfeld introduced the third heuristic of the day, "a more elaborate version of Pólya's strategy," alternately speaking and writing on the board.

> Suppose the problem asks for something, that's what I mean by a specific condition in the problem you want, Pólya says relax the condition, ask for something, ask for less. Since you're less demanding, there ought to be more solutions. There could be a whole family of them. So if you get a whole family of them, maybe you'll find the one you want in that family. [His written statement reads, "If there is a special condition in the problem you want, relax the condition—ask for less. Since you're less demanding, there should be a whole *family* of solutions. Look for the one you want among them."] [his emphasis]

Turning then to the issue of specific conditions, he asked, "What does the problem ask you for?" A combination of student suggestions and Schoenfeld's interpretation produced two conditions which were also written on the board: (1) the desired figure was a square and (2) all four of its vertices must lie on the square.

He chose to tackle the first condition, asking the class what was easier than a square. A student (invisible to the camera) responded quickly, "A rectangle." When Schoenfeld asked how a rectangle could be inscribed in the triangle he had drawn on the board, Devon began to outline a construction using three perpendicular segments. Schoenfeld accepted and completed his procedure, quickly producing three different rectangles, including a "short and fat" and a "tall and skinny" example, as reproduced below in Figure 6.

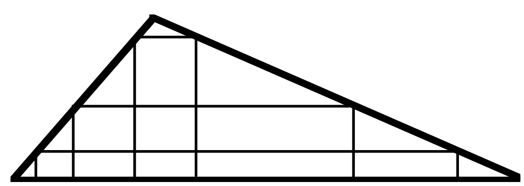


Figure 6
Three Inscribed Rectangles

From these three examples, Schoenfeld completed the continuity argument for the existence of the inscribed square: if the short, fat rectangle were transformed continuously into the tall, skinny one, that process must generate a rectangle with equal base and height (i.e., a square) somewhere along the way. He added that this was "actually the same continuity argument that I used before for the angle bisector." But with this existence proof in hand, he then denied the possibility of elaborating it into a constructive proof.

> Now that's a nice existence proof. *I don't know how to turn that into a constructive proof.* So that's actually argument number one, that's part of the problem but not the whole problem and to this day, *I don't know how* to take that nice, little existence proof and say, "Yeah, you can use that and, out of that here is a sequence of things you can do with straightedge and compass." [emphasis added]

He then turned to the second condition and asked for volunteers to generate squares with three vertices on the triangle that he'd drawn on the board. Three students, Sasha, Devon, and Stephen, came to the board, each producing an "inscribed" square of different size and orientation (Figure 7).

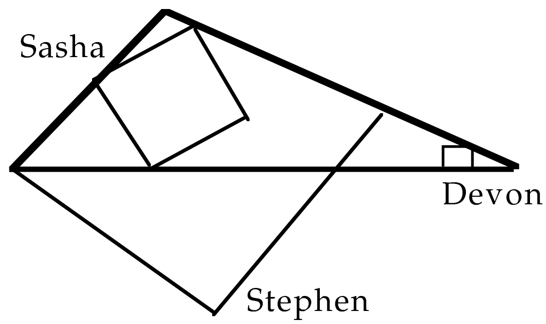


Figure 7
Three Students' Partially Inscribed Squares

Stephen's construction assumed the top angle of the triangle was a right angle. After noting this flaw, Schoenfeld erased that square, leaving two. He asked once more for other examples and, hearing no volunteers, noted his surprise that no one drew a square in the opposite corner to Devon's, because that "normally happens." He then drew increasingly larger squares with the same orientation as Devon's (see Figure 8) and asked, "What you can tell me about that family, what happens to the fourth corners?" Sharon responded, "They're given any range of sizes and then when you finally meet a distance where the, where the fourth one, where the fourth vertice meets the triangle, you have four equal sides."

Schoenfeld drew a squiggly locus connecting the fourth vertices of the squares (Figure 8) and restated Sharon's response in terms of the intersection of the locus and the triangle, thus completing the second existence argument.

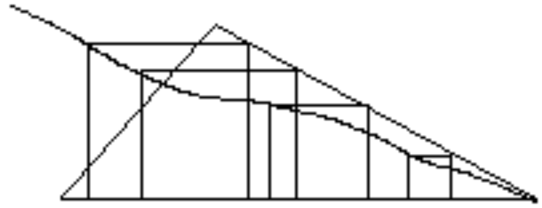


Figure 8
The Squiggly Locus Connecting the Fourth Vertices of "Inscribed" Squares

With the end of class approaching, he directed the class to "play with this example for Wednesday and see what you discover." This "assignment" seemed a clear indication that the second existence argument was more likely to generate the missing construction than the first.

The constructive argument (phase 4). At the start of the next class two days later Schoenfeld pointed the class back to the problem, "We left with problem 2 partly solved and partly up in the air. Does anyone have anything to say about problem 2?" When Devon volunteered and came to the board to show his solution, Schoenfeld turned to the class and described some of his longer-term goals for the class.

> One of things that I want to do during the course of the semester is get us talking like a mathematical community and ultimately using the standards of the mathematical community which means not like mumbling on the board, but instead being fairly clear, lucid, really making arguments clear so that all of us can understand precisely what's going on. So I'm going to push for those kind of standards in explanation, which means not just beautiful finished products, but also explanations of how and why it's reasonable that you did what you did and things like that. OK?

Figure 9 gives the location of the participants in the classroom at that point in Session 2.

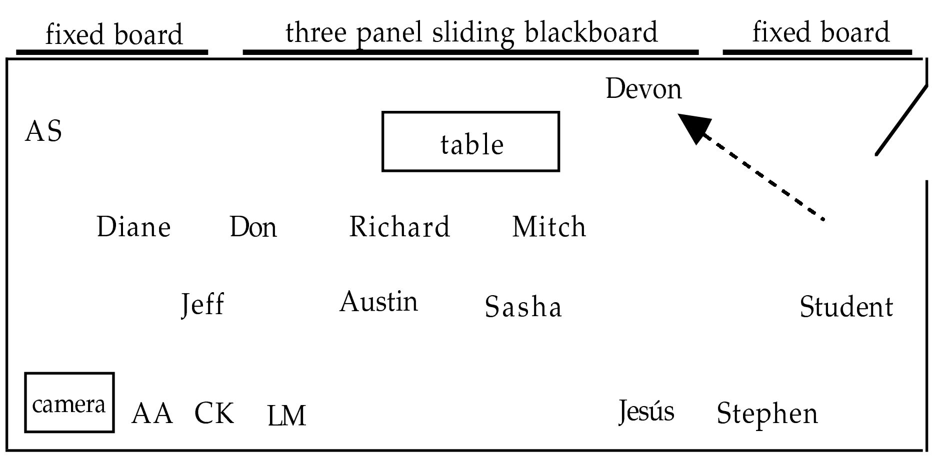


Figure 9
Arrangement of the Class: Start of Session 2
Devon's Construction of the Inscribed Square

Devon’s argument was based on his insight that the inscribed square could be produced by simply scaling another square up or down and therefore that the problem could be solved using similarity. He first explained how to construct a square with three vertices on the square and the fourth lying outside the triangle (Figure 10, frame 1).

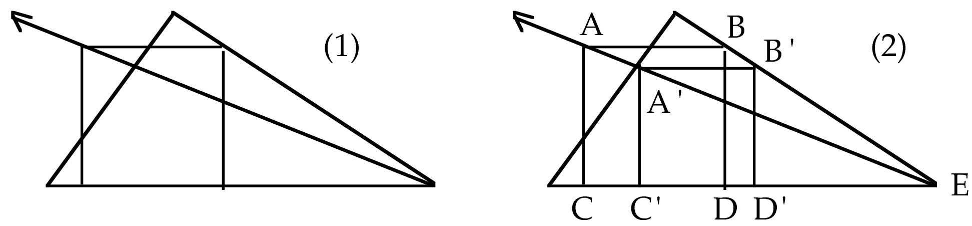


Figure 10
Devon’s Drawings Illustrating the Inscribed Square Construction

Then he drew the line from the far vertex of the triangle (point E, in frame 2) through the outside vertex of the square. (This line is the accurate representation of Schoenfeld's “squiggly locus.”) The point of intersection of that line with the triangle (point A' in frame 2) is one vertex of the inscribed square, and the rest of the square A'B'D'C' can be constructed from that point by dropping perpendiculars to the other two sides of the triangle. Devon went to show how, via similar triangles (e.g., ABE ~ A'B'E'), the fact that ABDC was a square guaranteed that A'B'D'C' was also.

Throughout his presentation, Devon faced and addressed himself primarily to Schoenfeld, not the class. Schoenfeld, still seated in the front corner of the room, called students’ attention to this phenomenon and suggested different standards for their presentations in class.

> A comment and a question: The comment is that what just happened in terms of [Devon’s] behavior is exactly what happens on the second day of class every time I teach this course, which is that: I ask someone to do something at the board and he spends 90 percent of his time looking at me for approval. I’m actually pretty good at playing poker and not revealing whether something is correct or not. I’m going to do that a lot, because ultimately I don’t *want* to be the judge of what’s right or wrong. The judge of what’s right or wrong in some sense is the mathematics and in another sense, it’s the class. And what I want this to be is a community that develops

> its own standards about mathematical correctness and it argues about when it buys an argument or not. So that's my comment. The question then is, did you guys buy what you've seen? Is that sufficiently compelling that you all believe the construction that [Devon] suggested? [pause during which Mitch says, "I'm not really convinced it's a proof."] I saw three heads nodding [Jeff's was one], I saw a bunch that didn't react.

Perhaps in response to Mitch's skepticism, Devon said that he "could argue for it." His restatement added more detail, e.g., ABE ~ A'B'E' and ACE ~ A'C'E' guaranteed that A'C' = A'B', but included no major changes. When he finished, Schoenfeld again asked the class if they were convinced, and this time no one spoke up. Devon went back to his seat, and Schoenfeld returned to the center board and explained how he would generally evaluate students' presentations. His evaluation of Devon's argument was different and explicitly positive, though with qualifications.

> One of the things that I'm going to do throughout the period of the course is ask nasty questions. Some of the times when I ask a nasty question that will indeed be true that it turns out that will be the case and that will knock your argument apart. Some of the times it turns out that your argument's right and I'm just being nasty [murmur of amusement from class]. That's because again the idea is what we're trying to do is make sure that the arguments are right. I buy this argument, and it needs a little bit of cleaning up maybe to be comprehensible to anyone who hasn't had [Devon] explaining it to them. But the structure of it, I think, is pretty nice and straightforward. It's still a little bit, looks like a rabbit pulled out of a hat, in that you have a nice explanation for something that's sort of presented there full-blown.

Then to connect Devon's proof to the previous day's line of development—and in particular to Relax a Condition—he showed how the proof could be interpreted in terms of relaxing conditions. Next came the standard "final" query, "Are we done?" A student (possibly one of the two students who entered the room during Devon's presentation) responded, "Think so." Schoenfeld replied,

> You'll learn within two weeks that's almost always a rhetorical question. The answer is "No" because there's still more we can do with that. Let me [erases board], let me return to a point where we left off on Monday and that

> actually will wind up with the same construction but might give a different idea of how it actually works.

He reconstructed a series of partially inscribed squares in a triangle, highlighted the fourth vertices, and asked what might be true about this locus. A student (not visible on the videotape) responded that it might be linear. Schoenfeld then left it for the class to verify that the locus was indeed a straight line.

Discussion

Of the many instructional issues Schoenfeld addressed early in the course, none was more important than the task of introducing heuristics so that students quickly appreciate their importance and gradually begin to use them intelligently. What does Schoenfeld's management of the solution of the inscribed square reveal about his approach to this teaching problem?

Before turning to the solution itself, it is important to consider the place of the inscribed square in the course as a whole. Two major components of Schoenfeld's curriculum are his problems and the heuristics that he introduces with them. If students are ever to see heuristic strategies as problem solving tools worth learning, they must face problems that do not easily submit to techniques they already know. In contrast to the two other problems discussed in detail in the first week, most students found it difficult to make progress on the inscribed square. Because that problem stumped most of the class, Relax a Condition could then demonstrate the power of heuristics. Likewise, Try an Easier Related Problem illustrated that one needs experience, skill and even patience in order to use heuristics—that they can't be applied in rote fashion. The discussion of the pitfalls of easier related problems helped to clarify their non-deterministic, "rule of thumb" character and show the importance of how you apply them. So Schoenfeld's solution of the instructional problem required demonstrably hard problems and quickly useful, if non-deterministic, heuristics.

But difficult problems and potentially useful strategies are only part of the story. Schoenfeld's extensive experience with the inscribed square problem provided well-grounded expectations about what students' likely responses would be (e.g., which easier related problems they would generate). These expectations complemented his knowledge of which solution paths would be more accessible to students than others (e.g., which existence proofs led toward constructions). This knowledge made it easier to recognize and interpret his students' suggestions and guide their efforts toward a successful conclusion.

These three top-level features (problems, heuristics, and prior experience with the problems) are all consistent with Schoenfeld's stated goals for the class. Solving problems (as opposed to exercises) supports his claim that the activity in and around the class reflects important aspects of professional mathematical practice. Learning to judge when, if, and how to apply particular heuristics is an important part of that practice. And the fact that extensive teaching experience with particular problems was central to using those problems productively reflects the complex relationship between problems and heuristics. But his management of the solution itself, particularly his appropriation (or not) of students' suggestions, bears a more complex relationship to his goals for the class.

It is important to recognize that the students played an active and substantive role in solving the inscribed square. Nearly half of the class contributed some piece of the evolving solution, and Devon's work, especially his construction, was more than simply "a contribution." Schoenfeld deliberately solicited their participation, but he also carefully organized and controlled it. Student input was solicited at certain points in the solution (e.g., when a range of possible approaches was needed) and not others, and their suggestions were assimilated into his instructional plan. His role as instructional leader and, at crucial junctures, mathematical authority was central to the pace and process of the solution. He orchestrated student participation within a relatively traditional model of roles for teachers and students, where teachers decide what choices to offer to students and when it is best to do so. These traditional elements of teaching appear—at first blush—to run counter to his stated long-term goal of creating a mathematical community where authority rests with the mathematics and community as a whole.

Before attempting to resolve this apparent contradiction, we review the evidence that undergirds these interpretative claims. First, what indicates that Schoenfeld came in with a pre-existing plan for solving the inscribed square? Though we did not question him at the time about his plans, he has written about his purposes for using the problem (to demonstrate the difficulties of applying heuristics), the easier related problems he expects to see (try a rectangle, try a circle, try a special triangle), the two existence proofs, and the difficulty of obtaining a construction from one of them (Schoenfeld, 1985, pp. 84–91). Most of the major elements of the solution that he orchestrated with the class are present in this written account. Given the strong similarities between the two, it is difficult to doubt the existence of a detailed instructional plan. Schoenfeld chose the inscribed square as the second problem of the course for particular reasons, and the main elements of the discussion were in place for him before the problem sheet was given to students.

This plan then became the framework for guiding the class through the solution and underscoring major points along the way. To achieve these goals in a reasonable time frame, he directed the class down certain pathways (and not others) and used his own mathematical experience and authority to justify these choices. As warrant for this interpretation, we summarize in chronological order the instances where Schoenfeld used his personal authority to direct the course of the solution.

- He wrote Devon's suggestion to seek a counterexample below the other suggestions and did not seriously consider it in the subsequent discussion.
- He delayed dealing with Mitch's suggestion to relax the condition on the square and try a rectangle (suggestion 1) until he discussed suggestions 2 and 3.
- He asserted that inscribing a circle in the triangle could not be adapted to inscribing a square in a triangle.
- He asserted that the solutions for isosceles and equilateral triangles were neither easy to produce nor to adapt to the arbitrary triangle.
- He asserted that the first existence proof could not be adapted into a constructive proof.

What can we learn about his teaching from these choices? First, given the match with his plan for the solution, each move is an example of how teachers appropriate students' ideas and suggestions to their own plans (Newman, Griffin, & Cole, 1989). Appropriation is a tool for balancing the dual goals of engaging students' interest and participation and sustaining progress toward important instructional goals. Second, while these instructional decisions were all explicit in the data, they were not all identical in character. The first two were management choices; they were decisions about what to take up for discussion and what to set aside, more than direct evaluations of what could be done mathematically. The first decision was sensible since the search for counterexamples would have been very difficult to assimilate into his plan. So Schoenfeld honored the suggestion in general terms and set it aside. The second decision was equally sensible since developing this suggestion first would have removed the possibility of teaching the lesson about the pitfalls of easier related problems. In that sense, Mitch's suggestion to relax a condition on the square was too good.

The last four, however, all involved Schoenfeld's explicit judgment of what is possible and productive mathematically and were justified by appeal to his own mathematical experience. Essentially they communicated the message, "Trust me. I have explored this problem extensively, and I know its 'ins' and 'outs'." Like the management of Easier Related Problems, these declarations sped the solution along, by curtailing potential solution paths that Schoenfeld knew to be unproductive. But to do so, he implicitly asked students to accept his role as the mathematical leader and decision-maker.

How then do these teaching moves fit with the overall goal of creating a mathematical community in the classroom? More generally, how do mathematics educators deal with mathematical authority, balance their informed authority against emerging student autonomy, and support students' growth toward a powerful and independent mathematical competence?

The first step toward a resolution is to acknowledge the mismatch: the solution of the inscribed square was inconsistent with some of the overarching goals of the course. Schoenfeld's directed solution does not easily square with the ideas of a classroom mathematical community pursuing its own solutions, and his statements about what was possible mathematically are not consistent with the methods of public justification employed by the professional community. On the other hand, the Overview points to evidence that Schoenfeld's teaching has moved students substantively toward his declared long-term goals, and our observations suggest this was the case for the 1990 class as well. Students became more proficient problem solvers; they learned to use heuristics productively; they interacted as a community of problem solvers; and they accepted the task of judging and nudging each other's ideas and arguments. So the question must be restated as, "Why was the direction not counterproductive to the long-term goals?"

Our view is that there were more important goals for Schoenfeld early in the course and that these may have required his exercising a leadership role in deciding some issues for the class. He must convince students that they have important things to learn in the course and that he will support them in that effort. If he did not strongly guide students' problem solving in the early days, they could easily flounder, pursue too many deadends, and come to question the entire enterprise. Instead, Schoenfeld made sure that they struggled enough to realize that they could not solve the problem easily but could be successful with his direction and the proper tools. These experiences were part of the transition toward more independent problem finding, problem solving, and justification. In short, his directive instruction gave priority to some goals over others.[9]

Our goal in emphasizing the complex relationship between instructional goals and teaching practice in this segment has not been to question or endorse the optimality of Schoenfeld's decisions. Rather, we have shaped the analysis to illustrate the interaction of ambitious educational goals, detailed instructional plans, teaching moves, and students' contributions. One main lesson is that innovative teaching oriented toward ambitious, non-traditional goals can embrace both traditional and non-traditional elements. The achievement of such goals may depend as much on the traditional elements as the non-traditional and on skillful balancing of short-term objectives and quite different long-term goals. This conclusion, we believe, undermines simple descriptions and explanations of successful non-traditional mathematics teaching. We need to look more closely to understand what works in these settings and why.

[9]We acknowledge that Schoenfeld did take some explicit actions toward building the classroom community and shifting the locus of authority in the first week, e.g., his statement to the class about standards for written and oral arguments before Devon presented his constructive argument and his statement about asking nasty questions to the class afterward. Our argument is that this was not his primary goal in the first week.

Practicing mathematical communication: Using heuristics with the magic square[10]

Cathy Kessel

The language is not alive except to those who use it.
(Thurston, 1994, p. 167)

Schoenfeld (1991) gives an account of a classroom discussion of the magic square problem. Here is the version of the third problem he gave to his students in the class.

> Can you place the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 in the box below, so that when you are all done, the sum of each row, each column, and each diagonal is the same? This is called a magic square.

In his account of the discussion of this problem Schoenfeld describes briefly how the problem, though trivial, can be used to illustrate many heuristics and other important aspects of mathematical thinking: Establishing Subgoals, Working Backwards, Exploiting Symmetry, Working Forwards, using systematic generating procedures, focusing on key points for leverage, exploiting extreme cases, solving a problem in more than one way, and using a problem as a springboard for further mathematics. At the end of his account he says that an important aspect of the discussion, the classroom dynamics which “reflected the dynamics of real mathematical exploration” was not described. One might wonder how a classroom discussion could reflect the dynamics of mathematical exploration and how such a discussion could happen on the second day of a course. The goal of this section is to examine the classroom dynamics of another magic square discussion, led by

[10] I would like to thank Alan Schoenfeld for the many ways in which he helped to make this article possible. His support, and that of the Functions Group and the School of Education at the University of California have helped me to learn about and do research in education. For comments, criticisms, and encouragement as this article slowly evolved, my thanks go to: Margaret Carlock, Marisa Castellano, Judith Epstein, and Jean Lave; the *RCME* editors: Ed Dubinsky and Jim Kaput; the *RCME* reviewers: Barbara Pence, Beth Warren, and one anonymous reviewer; Mary Barnes, Sue Helme, and Derek Holton; and my co-authors: Abraham Arcavi, Luciano Meira, and Jack Smith.

Schoenfeld in the fall of 1990, and to consider some of the features of mathematical practice it reflected.

Several aspects of this discussion are striking. Very little was written on the blackboard. What did appear were heuristics, diagrams, an equation, and a question, rather than the line by line theorems and proofs of traditional upper division courses or the line by line theorems, examples, and solutions of lower division courses.

The kind of speaking in Schoenfeld's classroom also differed from that of a traditional class. For example, in one seventeen-minute segment of the whole-class discussion of the magic square, though the teacher was at the front of the class and the students were not working in groups or presenting solutions, there were fourteen utterances—questions, comments, suggestions about the mathematics at hand—made by students. One would expect only questions in a traditional class and very few at that.

These differences suggest that communication is an important feature of this class and that research on language and communication might illuminate some of the reasons why Schoenfeld has chosen to conduct his class in this manner. In this section I use some analytic frameworks from sociolinguistics to describe classroom communication and to compare it with that of research mathematicians. I have been an undergraduate, graduate student, and faculty member at various mathematics departments across the U.S., and I draw on this experience as well as on written accounts to describe the teaching and research practices of mathematicians.

Differences in speaking and writing

At the beginning of the first day of a typical upper division undergraduate mathematics class, the professor writes her or his name, office location, and office hours on the board. After a brief statement about determination of course grades, she or he begins to lecture, starting with definitions and notational conventions and perhaps reaching some new material midway through the first class. There are few, perhaps no, questions from the students, which will be true for the rest of the term. The professor writes almost everything on the blackboard and almost everything has one of the following labels: theorem, lemma, corollary, proof, example, axiom, conjecture, definition, notation. One exception is pictures or diagrams, another (in applied mathematics classes) is applications. With such a lot of writing to do, blackboards, chalk, and erasers become extremely important. Professors become skilled at arranging their writing on the board, not erasing theorems or diagrams until they won't need to refer to them again. (An inattentive professor may also cover one sliding board with another so quickly that students can't finish copying the writing on the board. Schoenfeld did this on the first day of class in his enactment of the typical calculus professor, as described in Meira's section.)

The main focus of the typical mathematics class is the blackboard and students' main activity is taking notes, and following the lecture. As in written mathematics the statement of results tends to be impersonal. Names occur mainly in important theorems, definitions or axioms, e.g, Stokes' Theorem, Green's Theorem, Gödel's Incompleteness Theorem, the Zermelo-Frankel axioms of set theory, the Peano Postulates, Noetherian ring, Abelian group. The time at which the object was constructed isn't often mentioned. (Later, as students reach the edges of mathematical knowledge in graduate school, names and dates appear with much greater frequency.[11])

The classroom language of the typical mathematics professor reflects the way mathematics is presented in writing. "Assume the following holds . . . ," "it follows easily that . . . ," "it is not the case that . . . " are frequent phrases (for more examples see Pimm, 1987). Rotman (1993, p. 7) describes written mathematics as "riddled with *imperatives*, with commands and exhortations such as 'multiply items in *w*,' 'integrate *x*,' 'prove *y*,' 'enumerate *z*'" and "completely without *indexical* expressions, those fundamental and universal elements of natural languages whereby such terms as 'I,' 'you,' 'here,' 'this,' as well as tensed verbs, tie the meaning of messages to the physical context of their utterance." For example, the magic square problem could have been stated in the following way: "Place the numbers 1 through 9 in the boxes to the right so that the sum of each row, column, and diagonal is the same. Such an arrangement is called a magic square." This imperative statement doesn't mention where and when the action of placing numbers in boxes is occurring, nor who is acting.

[11]This description is not meant as a condemnation. Some of my happiest and most instructive hours have been spent in such courses. I appreciate well-tended blackboards and good chalk and my spoken language reflects written mathematics. At the end of this section I suggest some reasons why not all students are so fortunate.

Chafe and Danielewizc (1987) characterize a style with few indexical expressions as "detached," "show[ing] an interest in ideas that are not tied to specific people, times, and places, but which are abstract and timeless" and which avoids mentioning concrete doers. They note that this style predominates in writing and suggest a reason for it—"for writers, the audience is usually unseen, and often unknown" (p. 19). This detached style of speaking and writing about mathematics suggests to listeners and readers that mathematics is independent of time and place. This is consistent with the epistemology (a mixture of formalism and Platonism) held by many mathematicians (Davis, 1986; Davis & Hersch, 1986; Ernest, 1991; Fauvel, 1988). It is also consistent with the epistemology held by many high school students that learning mathematics is mostly memorizing facts (National Center for Educational Statistics, 1993) and that the ideas of mathematics have always been true and will always be true and were discovered (not invented) by mathematicians (Clarke, Wallbridge, & Fraser, 1992). And it is also consistent with the mathematics classroom experiences of undergraduates (Mura, 1995).

In contrast, note Schoenfeld's wording of the magic square problem which begins, "Can you place" rather than "Place the numbers." We shall see many indexicals—"I," "we," "you," "Devon's question," "what Jeff guessed last time"—in Schoenfeld's speech. Chafe and Danielewizc call a style with many indexicals "involved" and note that it is characteristic of, though not limited to, spoken language. They also note, "In most spoken language an audience is not only physically present, but has the ability to respond with language of its own" (pp. 18-19). This suggests that, in addition to acknowledging their presence, an involved style may invite listeners to respond.

And Schoenfeld does want his listeners to respond. Because a main goal of his course is the creation of a "mathematical community," one of his goals for the first days of the course is to get the students to talk: "Clearly what I need to do is begin pulling things from [the students] because part of what the course is supposed to do is turn things over to them" (audio taped discussion, May 22, 1991).

Schoenfeld's language is not only involved, but informal and non-technical with occasional shifts in style to language that reflects written mathematics. Such shifts are known as code-variation. Saville-Troike (1989) defines codes as "different languages, or quite different varieties of the same language" and code-variation as a change in code within a speech event. She notes that code-variation may serve many different purposes, depending on context. Here, Schoenfeld's shift to language reflecting written mathematics serves to display his group affiliation as well as to help the students to become familiar with mathematical language. Just as patients don't have faith in a doctor who "doesn't talk like one" despite the miscommunication that can occur when doctors use technical language (Saville-Troike, 1989), students may not have faith in a professor who doesn't talk like one. Schoenfeld must talk a fine line between being understandable, approachable, and interested in students' contributions, and maintaining his status as the knowledgeable member of the mathematics community depicted in his monologue at the beginning of the first class session. His informal language may suggest to students that it is acceptable if they reply in the same manner. It also contrasts with and emphasizes the few technical words he does use: names of heuristics (which he sometimes labels "jargon") and mathematical terms.

As with spoken language, there are differences in how much Schoenfeld writes, what he writes, and the way he uses writing. This difference suggests a different emphasis, and a different view of what is important in this classroom. Not only does Schoenfeld write considerably less than a traditional mathematics professor, but when and what he writes are different. Both mathematics and heuristics appear on the blackboard as one would expect in a class on heuristics and their use. The diagrams in this class would not be seen in a textbook or an article, they are used to work with, rather than to illustrate. They are altered throughout the course of the discussion and erased only when the discussion is over. This use of diagrams allows Schoenfeld to avoid technical language as well as to make the problem more immediate and his descriptions more direct. The other blackboard writings are names and brief descriptions of heuristics. The connection between mathematics and heuristics is not recorded on the blackboard, it is made through questioning, and interaction with the class. Formal proofs aren't given, instead, their genesis is enacted in the classroom discussion. Later in the term, students will present their own conjectures and proofs.

Not only are there differences between the informal, involved style of his language and the detached style of a traditional professor of mathematics, there are also differences in the content of Schoenfeld's language and its mode of use. Its content includes what is traditionally thought of as the subject matter of mathematics classes, but is also *about* math, about how math gets done, about revealing "the tools of the trade," and about learning that trade. This suggests a different emphasis—not only is mathematical content important, but how one does mathematics is a legitimate topic of classroom discussion.

As Arcavi points out in the overview, Schoenfeld has two different modes of communicating with the class as a whole: presenting and involving the class in discussion. As we have seen in the previous sections and shall see in this section, Schoenfeld uses these different modes of communication for different purposes. In general, Schoenfeld presents heuristics, either as an explanation of a mathematical suggestion given by a student, a name for a process that's just been illustrated, or to give direction to the solution of a mathematics problem. In the latter case, the heuristic is frequently instantiated in the mathematical context at hand by the students.

Schoenfeld gets the students to instantiate heuristics by a method of questioning similar to that described by Pólya (1973) in *How to Solve It*.

> The teacher's method of questioning . . . is essentially this: Begin with a general question or suggestion on our list [of heuristics], and, if necessary, come down gradually to more specific and concrete questions or suggestions till you reach one which elicits a response in the student's mind. . . . It is important, however, that the suggestions from which we start should be simple, natural, and general, and that their list be short. . . . The suggestions must be general, applicable not only to the present problem but to problems of all sorts. . . . The list must be short in order that the questions may be often repeated, unartificially, and under varying circumstances. . . . It is necessary to come down gradually to specific suggestions, in order that the student may have as great a *share of the work* as possible. . . . Our method admits a certain elasticity and variation, it admits various approaches . . . it can and should be applied so that questions asked by the teacher *could have occurred to the student himself.* (pp. 20-21)

Schoenfeld has taught his class many times before. Though he does not know his students well at the beginning of the course, as in the cases of the telescoping series and the inscribed square, he knows most of the responses students will make to his questions about the magic square. Students are not completely predictable though and Schoenfeld's management of the discussion also had opportunistic elements (Hayes-Roth & Hayes-Roth, 1979; Schoenfeld et al., 1992). Students' questions and suggestions, both predictable and unexpected, were used to serve goals of the discussion and of the course.

The nature of the problem

Both the magic square problem and the way in which it is used are important elements of this discussion. The magic square was the third problem on the sheet given to the students and the third to be discussed. One difference between the magic square problem and those preceding it is that its solution is indubitable (it can be checked by a simple calculation), and easy to reach. Unlike the problem of inscribing a square in a triangle, it is easy to solve, and couldn't be used to convince the students that they needed to learn heuristics. It may even suggest to students that they have been using "raw" heuristics—heuristic tendencies that need refinement before they are likely to be consistently useful (Silver, 1985).

However, the magic square serves as an excellent vehicle for the introduction and illustration of heuristics. Because the mathematics involved is elementary, students can discuss it without the fear of displaying ignorance—it's easy to talk about (Schoenfeld, audio taped discussion, May 22, 1991). Because there is no need to focus on getting a solution, students can focus on the process of arriving at a solution. What follows is an account of the discussion of the magic square. Annotations and interpretations in brackets are interspersed. Italics is used in two different ways in the transcript: Italicized phrases were both spoken and written on the blackboard; single words in italics indicate words emphasized by the speaker.

The discussion of the magic square

Schoenfeld asks for volunteers to present the magic square problem. Jeff volunteers and presents his group's solution. After Jeff sits down, Schoenfeld goes back to the board, acknowledges the solution ("the answer speaks for itself") and indicates a transition to another activity "What I want to do is play with this a little bit. First of all it's not a problem you want to do by *pure* trial and error." He then gives a standard combinatorial argument to show how many different ways there are of filling a 3 by 3 grid if one places the digits from 1 through 9 randomly in its cells. He notes, "There are 9 ways that you can stick any number, say [points to top left square of grid], in this square, 8 in that one [points to top middle square] after you've used one, 7 in the next one" and so on. This yields 9! ways to fill in the 3 by 3 grid—but, he observes, the magic square has eight-fold symmetry, so there are $\frac{9!}{8}$ = 9 x 7 x 6 x 5 x 4 x 3 x 2 x 1 non-equivalent ways one might fill it. Then, saying intermediate products aloud, he quickly calculates the result.

[Professors teaching undergraduate courses don't often do computations involving large numbers in front of their classes. One reason for doing it here might be to emphasize the improbability of obtaining a solution by random search. This will contrast with the solutions obtained by the use of heuristics that will follow. Another might be to display mathematical expertise to those who consider quick calculation a sign of expertise. A more subtle message which this calculation may convey is that none of the students used pure trial and error. They all solved the problem.]

During his calculation Schoenfeld mentioned the eight-fold symmetry of the square. Devon asks if there were any solutions to the magic square not equivalent using symmetry. Schoenfeld replies "That's a good question, let's leave that as something to look at" and writes the question on the sideboard where it remains for the rest of the discussion.

[Schoenfeld's knowledge of the magic square and its different solutions allows him to make this response, knowing that the question will be answered before the class ends. His action serves several purposes: It legitimizes the student's question without immediately changing the flow of the activity, begins a community history and gives an example of mathematical practice—questions are important, they may not be immediately answered, but one may later note as Schoenfeld will do, that a particular question has been answered by a proof or construction.]

Schoenfeld says "So if you don't want to do it by trial and error then what you really want to do is look for ways to reduce the number of things you've got to consider" and summarizes Jeff's presentation of group's work as a strong appeal to symmetry " . . . if you make those two guesses, 5 is in the center and 15 is the sum then you don't have too much trial and error to do before you get there. And that's a good sane way to go about doing the problem." [Here the students may be reassured, they all solved the problem, and their solutions weren't gotten by pure trial and error. The "two guesses" will reappear as instantiations of heuristics, again suggesting that the students may be using "raw heuristics" which can be refined.]

First (re)solution: By establishing subgoals and working backwards

First subgoal: What is the sum? So far heuristics haven't been mentioned. Schoenfeld shifts the focus to heuristics, says "What I want to do is ask a couple of questions that illustrate some of Pólya's strategies and use the answers to make progress on this problem again so we're going to revisit the problem a little bit." He erases the board and states "We're back to the beginning, we want to place the digits from 1 to 9 into this [the empty grid he has just drawn] so that the sum of each row, column, and diagonal is the same." Now he introduces a heuristic. [Note the shift in the meaning of "you" in Schoenfeld's first utterance. At the beginning "you" is an unspecified person, perhaps one of the listeners. At the end it is Richard.]

AHS: The first question is generic: *What piece of information would make the problem easier to solve?* [He turns to face the class.] That's a really broad generic question. But you're facing a problem, it's posed in a particular way. Now you can ask yourself is there some piece of information, some bit of knowledge, so that if you just had that, the problem would be significantly easier to solve? [To Richard] And you're nodding your head yes, what would it be? [Schoenfeld moves closer to Richard.]

Richard: Just to look around for the sum of the triples . . . and add the three smallest numbers for the minimum, the three largest for the [inaudible].

AHS: OK. So the key piece of information is, or certainly a key piece of information is: this says that the sum of each row, column, and diagonal should be the same, it would be awfully nice to know what that number is, so what is the sum? [He writes "What is the sum?" on the board.] And we had a suggestion about how to think about this that I'll mention in a second. Let me throw some more jargon at you. This is called, as simple as it seems, in other contexts it's a little bit more complicated, and worth having a name, *Establishing Subgoals*.

Now Schoenfeld starts on the work of answering the question What is the sum? by noting easy upper and lower bounds on the magic number—it must be less than the sum of the largest three numbers in the magic square, 9, 8, and 7; and larger than the smallest three 3, 2, and 1.

He invites a response from the class by saying [33] “Is there anything else I can say about that sum?” Gary[12] responds. He seems to assume that Schoenfeld is considering a magic square with 1, 2, and 3 in a row, column, or diagonal because he says “You can narrow it even closer because if you used 1, 2, 3 in a single column, row, or diagonal then you know that you’re going to be building something even larger, because 2 and 3 for instance are already gone so you have to use 4, 5, and 6.” [Here Gary responds to Schoenfeld’s use of “I” by using “you” both of which suggest that Schoenfeld is engaged in doing mathematics, rather than presenting a finished product. Gary’s use of “you” also suggests a collegial relationship with Schoenfeld.]

Rather than clarifying his earlier statements, Schoenfeld rephrases Gary’s response: “OK, so in some sense the very least I can get for a sum if somewhere I’ve used 1, 2, and 3 in a row, that might try this row, that row, or something like that, the 3’s going to be involved in another sum, and that’s going to use at least 4 and 5.” He writes in the empty grid

1		
2		
3	4	5

saying, “And if that uses 4 and 5 . . . [his voice trails off and he pauses] What else can I say? [pause] This says that there’s going to be one sum that’s at *least* 12. [pause] Can you say anything else?”

During the next utterances the square on the blackboard undergoes the following changes (an arrow indicates that the square to its left has been altered to yield the square on its right):

1		
2		
3	4	5

→

1		7
2	6	
3	4	5

→

1	6	7
2	8	
3	4	5

[12]This student did not stay long in the course. He does not appear in our overview of students.

Gary: If you actually wanted to build it this way then you'd go up on the right with 6, and 7 next.

AHS: Well, that's good, you go 3, 6, and 7. Is the argument now that every sum has to be at least 16? That's what it looks like we just proved, right? No matter what magic square you draw, you're going to get one sum that's going to add up to 16? [pause]

Diane: No, because you could put the 3, 6, and 7 after the 1 [inaudible].

AHS: So the claim is, well I could put the 6 and 7 after the 1, that gives me a 14, but then I've got to use an 8 and that says now I've got a proof that I get at least a 16—a 17. [pause] What's happening here? [pause] We already saw that there's a magic square with a 15, but it looks like we just proved that you've got to get an 18. [pause] What's happening?

Gary: Well, we know that we can't have 1, 2, and 3 in the same line anyway because we can't construct a magic square from it.

AHS: [confidently and quickly in contrast to his previous utterances] OK. What we just showed is if you start with a 1, 2, and 3 in a row then you're going to get some fairly large sums, that doesn't mean that every sum has to be that way. OK. [Erases square.] So the sums are going to be larger than 6.

[Gary's line of inquiry, trying 1, 2, and 3 in the same column to begin a solution to the magic square, was not quickly curtailed as in the inscribed square discussion, though it also does not lead to a solution. In fact Schoenfeld encourages Gary by writing his suggestions on the board and asking "What else can I say?" (though in a rather uncertain tone of voice). This path is curtailed in an obvious sense by Schoenfeld, he erases the magic square and changes the subject. However, in contrast to the dead-ends in the inscribed square discussion, a student is involved as a collaborator in this action; Gary has noted that the assumption that 1, 2, and 3 are in the same column of a magic square can't be true.]

He asks "Is there any other way we can get a handle [on this] besides good guessing? And I don't at all, want to put good guessing down, a symmetry guess is an excellent way to go. Is there any other way we might get a handle on what this might be?" Devon responds that the sum of the three rows of the magic square must be equal to the sum of the numbers from 1 to 9. He then shows, using the grid, how its use might give rise to Devon's answer, and continues to show how the observation yields a proof that the magic number is 15. The subgoal has been achieved.

[Devon's suggestion has provided a "natural" way to introduce Working Backwards and to give an example of illuminating the source of a mathematical idea—showing that Devon's suggestion is not a "rabbit pulled out of a hat." Schoenfeld's use of this suggestion to introduce Working Backwards, a heuristic which he would bring up during the magic square discussion in any case, is an example of the opportunism described by Hayes-Roth and Hayes-Roth (1979).]

Second subgoal: What goes in the center? This is a natural moment to again invoke Establishing Subgoals since finding the center of the magic square is a useful next step. Schoenfeld pulls down the board with Establishing Subgoals and erases all but Establishing Subgoals. [Here is an example of traditional blackboard expertise and evidence of Schoenfeld's plan for this discussion.] He says,

> Since I have this statement, Establishing Subgoals, in a nice box on the board, why don't I take advantage of it again. We now know that the sum of each row, column, and diagonal is supposed to be 15. What's the next major piece of information that would help me make significant progress on this problem?

He again uses Pólya's method of questioning. Student 1[13] responds "What goes in the center." Schoenfeld answers "Yeah. What goes in the center" and presents another heuristic, Consider Extreme Cases. He then gives its mathematical instantiation in this context.

AHS: So let's ask an extreme case, can 9 go in the center of the square? That's as extreme as you can get. [He writes 9 in the center of the square.]
Student 1: No.
AHS: Why not?
Student 1: You run out of numbers that you can add pairs of to 9.

[13]This student was invisible to the camera and can't be identified with certainty. His voice appeared to be coming from the right side of the room.

AHS: If the magic number is 15, that raises a serious problem, where's 8 going to go? If I put an 8 there [he writes 8 in upper left corner] I need a –2 over there and I ain't got none. If I put an 8 there [the upper middle square], I need a –2 over here and so on. OK? So 9 *can't* go in the center. [He erases 9.]

[Here the code-shift from "that raises a serious problem" to the attention-getting "I ain't got none" emphasizes the reason why 9 can't go in the center of the square and mirrors the student's rather awkward sentence. Writing and erasing serve to dramatize what Schoenfeld is saying and to display his reasoning.]

He continues more and more quickly through the cases of 8, 7, 6, gets to 5, says "Maybe. How about the other extreme?," writing 1 in the center of the square. He continues and eliminates the remaining possibilities.

The subgoal of finding the center has been achieved. Schoenfeld doesn't point this out explicitly but makes the transition to the next activity by saying,

> Having gotten that far we could consider some trial and error. But we ought to at least take advantage of symmetry to see how much trial and error we really have to do. So let me ask the question, how many different places are there that we might stick a 1? There are really only two different places . . . [corner and side pocket].

He then explicitly shows the symmetry he has mentioned several times, using hand gestures accompanied by his verbal description of rotating the board. [The gesture of rotating the board is an example of a deictic (McNeill, 1992). It is a visual analogue of Schoenfeld's involved language: he, not some undescribed mechanism, is the rotater of the square. His gestures also allow him to give a definition of symmetry without using technical language.]

> There are really only two different places. If I had a solution with a 1 over here [writes 1 in the upper left corner] then—and all the rest of these were filled in, I could take that solution [puts his right hand, crooked left, over the center of the grid and straightens his wrist], take the board, and rotate it 90 degrees [he puts his right hand above the grid, his left below, and rotates them about the center of the grid so that the left hand ends above the grid], that gives me a solution with 1 over here [points to upper right corner]. Or

> equivalently, if I had a solution with 1 in the corner over here [points with his left hand to upper right corner], rotating it that way [his left hand moves up and to the right as he reverses his previous rotation gesture] gives me a solution with the 1 over here [points to the 1 in upper left corner]. Same for the other corners [points to the two lower corners]. So a solution with a 1 in the corner is equivalent to, or generates a solution with a 1 in any other corner. Similarly for 1 in a side pocket. That generates any of these [pointing to each side pocket in turn].

Another heuristic is quickly noted (it's just been illustrated); Schoenfeld writes Exploit Symmetry, saying "That's another strategy that comes in handy" and returns to the work at hand. He writes 1 in the corner of the square, notes that 9 must go in the corner diagonally opposite, and discusses the placement of 2. Using symmetry one need only check three places. Schoenfeld indicates each and shows that no matter where 2 is placed some row, column, or diagonal will not add up to 15. He concludes "So what I've just showed is there's no solution with a 1 in the corner. That leaves us a 1 in the side." He erases 1 and writes 1 in the side pocket, discusses the placing of 2 and finishes the solution. [Looking at the case where 1 is in the corner first makes the discussion smoother since this case doesn't hold. Such a choice is usual in both classroom and professional mathematical presentations.]

Now he summarizes, "What we've proved along the way that the 1 has to go in the side pocket, the 2 has to go in one of the two bottom positions opposite, and the rest is forced, so the answer is that's the only solution modulo symmetry, which answers Devon's question." [A mathematician who solves a problem posed by person X will, especially if X is famous, frequently say in an account of the solution "This answers a question of X." The episode of Devon's posing of a problem to its solution outlines in miniature the way a problem is posed and solved amongst professional mathematicians.]

Second (re)solution: By working forward

Schoenfeld asks the ritual question "Are we done?" and Jeff replies "We're never done." (The students are beginning to internalize the new classroom rituals.) Schoenfeld replies "You're learning" and makes the transition to a new activity "What I want to do is to go back to this problem in an entirely different way," summarizing the approach used before, erasing the board, then giving a description of the new approach which he's termed (1991) Working Forward. Here he doesn't label it, just describes and gives an instantiation of this approach—listing triples whose sum is 15.

Now he initiates the students' participation by asking for triples. Different students call out responses hastened by Schoenfeld's "Any more?" or "Another one?" which follows quickly after he writes each triple on the board. He lists 159, 294, 258, 168, 357, and 195, says "Oops, we got that already" and crosses it out. They continue 456, 762, and stop. Schoenfeld says "Are we done, is that all of them?" A student[14] produces 834 and Schoenfeld asks "Are there any more?" No one replies.

Schoenfeld tells the class, "This is now something like the 142nd time I've used this particular problem, 142nd time I've asked this particular question, "Are there any more?," and I get to ask the same next question for the 142nd time: How the hell would you know? You've sort of generated them randomly, so you got a whole bunch of them, but you might of caught them all and you might not."

[Here again code-shifting serves several goals: "hell" emphasizes the seriousness of the students' dilemma, and "generated" and "random" suggest mathematical affiliation. As he says to the students, Schoenfeld has seen classes implement his suggestion of listing triples of numbers unsystematically before. Here again, in contrast to the inscribed square discussion he's allowed, in fact encouraged, the class to follow a path which will not easily lead to a solution. The nature of the magic square makes this dead-end more quickly reached and more obvious than those occurring in the inscribed square discussion. One reason for doing this is to show the students that they're in need of his teaching as well as the heuristic, Be Systematic. Another is to illustrate the issue of control, the students don't know how to implement his suggestion in such a way that they know when they've achieved it.]

He mentions the strategy whose omission led the class into its predicament, writing "IT HELPS TO BE SYSTEMATIC!" on the side board. He summarizes the difficulty, pointing to the crossed out triple 195 which serves as a record of the students' activity and suggests a way to instantiate the strategy—listing the triples in increasing order and beginning the list with all the triples that start with 1.

[14] It was difficult to identify the students who participated in this segment, but possible to tell that different students were calling out triples.

To start this new path he erases the unsystematic list of triples from the board and starts the new list with 1 5 9. The class calls out the rest of the triples. Schoenfeld points out the connection between the triples and the magic square,

> So we've got a total of eight triples, . . . , that's nice, because there are eight rows, columns and diagonals. Now what was the most important square? In the magic square? The middle. How many sums was that square involved in? [Here he uses the empty magic square to calculate, drawing horizontal, vertical, and diagonal lines through the middle square to show that it is involved in four sums.] How many digits appear four times? Only the 5, that's the only digit that appears four times. So if there's a solution: Guess what, this is a completely independent proof, 5 has to go in the center square. [He writes 5 in the center.]

He uses this idea to show where the other numbers in the magic square must be placed—numbers which appear in only two of the listed triples must go in "side pockets" and numbers which appear in three of the listed triples must go in corners.

The finding of two solutions has been enacted.

> Now we've beat it to death. Are we done? [He pauses and looks at the class.] Of course not, because so far we've only solved the problem I gave you. If that's how mathematics progressed, mathematics wouldn't progress. Solving known problems is not what mathematicians get paid for nor is it anything they have fun doing.

Schoenfeld's closing statement illustrates some themes of the course, that problems may have multiple solutions and that solving a given problem is only the beginning (problems can be generalized, extended, etc.). This is one of the aspects of the course that reflects mathematical practice (Kitcher, 1984). After Schoenfeld's statement, students suggested extensions and generalizations of the magic square. In session 3, the class discussed ways of generating 3 by 3 magic squares with entries other than the numbers from 1 to 9. In session 4, after finding there are no non-trivial 2 by 2 magic squares, the students conjectured that there is no even-dimensional magic square, Mitch discussed a procedure for generating a 5 by 5 magic square, Christina[15] described a procedure for generating one of odd dimension, and Devon showed that the magic number of a 3 by 3 magic square is always divisible by 3 and that the number in the center is always one third of the magic number. This was followed by a discussion of what a magic cube might be. In session 9 Schoenfeld provided a counterexample to the conjecture that no non-trivial magic squares of even dimension exist by showing the students an engraving of Dürer's *Melancholia* which depicts a 4 by 4 magic square.

Why teach a class this way?

Implicit in the preceding description is the question of why Schoenfeld chooses to conduct his class in the manner he does. A proof that the magic square has a solution that is unique modulo symmetry could have been given in far less time. Why do it this way?

I'll begin with the issue of blackboard writing. The writing that appeared on the blackboard was devoted to names of heuristics, diagrams, an equation, and a question. One might consider Schoenfeld's blackboard writing to be in conflict with traditional mathematical practice, since it differs greatly from the kind of writing seen in textbooks and articles and in other mathematics classrooms. It was, as Schoenfeld says, "sparse and sloppy" (audio taped discussion, March 8, 1991) while that of textbooks, articles, and traditional mathematics classes is profuse and precise.

Certainly writing is an important part of mathematical discourse. However, its relationship with the way mathematics is done is not obvious to those who aren't mathematicians. Mathematicians' descriptions of mathematics show that writing is but one way of communicating mathematically. Davis and Hersh's (1986) Ideal Mathematician communicates results to fellow experts "in a casual shorthand" but in published writings "follows an unbreakable convention: to conceal any sign that the author or the intended reader is a human being." Stewart (1993) points out that,

> Much of mathematics is communicated by informal discussions over coffee, seminars, lectures, and other media that do not produce permanent records. When important mathematical ideas are "in the air," other mathematicians

[15]Christina was a computer science major who entered the class in the third session.

> get to hear of them by these informal routes, long before anything appears in a technical journal. (p. 121)

Thurston's description of mathematical communication gives a sense of the differences between spoken informal mathematics and formal written mathematics.

> One-on-one, people use wide channels of communication that go far beyond formal mathematical language. They use gestures, they draw pictures and diagrams, they make sound effects and use body language. Communication is more likely to be two-way, so that people can concentrate on what needs the most attention. . . . In talks, people are more inhibited and more formal. . . . In papers, people are still more formal. Writers translate their ideas into symbols and logic, and readers try to translate back. (1994, p. 166)

The description of the genesis of a proof by De Millo et al. (1986) suggests that written mathematics is the end of a long process which begins with informal communication.

> In its first incarnation, a proof is a spoken message, or at most a sketch on a chalkboard or a paper napkin. That spoken stage is the first filter for a proof. If it generates no excitement or belief among his friends, the wise mathematician reconsiders it. But if they find it tolerably interesting and believable, he writes it up. After it has circulated in draft for a while, if it still seems plausible, he does a polished version and submits it for publication. If the referees also find it attractive and convincing, it gets published so it can be read by a wider audience. (p. 272)

This aspect of mathematics is generally hidden from students (Rogers, 1992). One doesn't often see the genesis of a proof in a classroom, instead one sees the end-product of the process described above, presented in detached language that erases its author and origins. Such classroom experiences help to explain why students' ideas about the nature of mathematics are sometimes so very different from those of mathematicians. Some students may not even believe that mathematics is done by human beings (Belenky et al., 1986) just as some city children used to believe that milk grows in bottles. Work in cognitive science has shown that students' beliefs about the nature of a subject may have profound effects on their learning of it (McLeod, 1992). De-emphasizing writing and formal mathematics, not only reflects mathematical practice, but may also change students' beliefs about the nature and the doing of mathematics.

De-emphasizing writing and formal mathematics may have other consequences for students' learning. Thurston suggests that familiarity with the ideas of a subfield of mathematics may need to precede the ability to recognize the same ideas in written form.

> People familiar with ways of doing things in a subfield recognize various patterns of statements or formulas as idioms or circumlocutions for certain concepts or mental images. But to people not already familiar with what's going on the same patterns are not very illuminating; they are often even misleading. The language is not alive except to those who use it. (p. 167)

In general, students of mathematics, like those new to a subfield of mathematics, are not already familiar with what's going on. Mathematicians who want to learn about a subfield usually ask what the ideas, questions, and objects of that subfield are. Unlike mathematicians, students may not know to ask those questions and to look for idioms and circumlocutions in written mathematics. When they encounter written mathematics they may be focused on its form rather than its meaning; reading each line of a proof, rather than trying to understand the ideas behind it. More importantly, they may not have any sense that such ideas exist.

This suggests that an emphasis on formal written mathematics causes difficulty for students, both from a cognitive and a metacognitive perspective. Students don't appear to perceive and interpret formal mathematics as mathematicians do.[16] Their beliefs about the way mathematics is done and hence how they should learn mathematics are derived from presentations of finished products. It seems unlikely that their beliefs could be changed by seeing even more formal mathematics, particularly since the students' means of interpreting that formal mathematics would have to be addressed at the same time. Biographies of many mathematicians suggest that informal mathematical experiences, often occurring outside of the classroom, were an important factor in their mathematical development (see for instance, Albers & Alexanderson, 1985; Albers, Alexanderson, & Reid, 1990; Hersh & John-Steiner, 1993; Ulam, 1976; Weil, 1992).

[16]In cognitive science terms, students don't appear to have schemata for formal mathematics similar to those of mathematicians.

Thurston's statement, "The language is not alive except to those who use it" and Schoenfeld's (1994) statement, "When mathematics is taught as dry, disembodied, knowledge to be received, it is learned (and forgotten or not used) in that way" outline extreme cases which might illuminate the problem of how to teach students mathematics. What I have tried to suggest in analyzing the language of Schoenfeld's classroom discussion, is that the discussion is an example of embodying mathematics—presenting it as a particular kind of communication to be used by all the people in that classroom, rather than as knowledge to be learned. The complexity inherent in the word "communication" is suggested by listing some of its components (Saville-Troike, 1989): linguistic knowledge; interaction skills (this includes perception of salient features and norms of interpretation); cultural knowledge (this includes values, attitudes, and schemata). In this view, transmission of knowledge and skills is just one aspect of communication. Similarly, communication among mathematicians is not restricted to formal writing, it includes other methods: informal writing, as well as talking, gesturing. Moreover, values, attitudes, and schemata are an important part of mathematical communication. The work of Schoenfeld and others suggests that these other aspects of mathematical communication play an important role in students' learning of mathematics. Schoenfeld's classroom suggests that such aspects of mathematical communication may be taught inside as well as outside the classroom and thus, unlike me and many other mathematicians, students need not wait until they begin doing research to start communicating mathematically.

Concluding discussion

In this section we synthesize our analyses of the early stages of Schoenfeld's problem solving course and offer some implications. We began this article with an illustration of some long-term goals of Schoenfeld's problem solving course: That the class become a "mathematical community" advancing and defending conjectures and proofs on mathematical grounds; and that the locus of authority be the "mathematical community," not the teacher. Because students' experiences in mathematics classrooms are, in general, very different from those of the community he wishes to create, achieving these goals is not easy and the path from the beginning of the course to a microcosm "of selected aspects of mathematical practice and culture" (Schoenfeld, 1994, p. 66) is not obvious. Rather than examining its later stages when its beginnings were likely to be invisible, we focused on the course at its inception. Our initial question was: How does Schoenfeld create a community of problem solvers where undergraduates learn to think and *do* mathematics, when their past experience in mathematics has mainly involved listening, writing notes, and learning procedures?

Short-term goals

After twelve or more years of schooling, undergraduates usually have well-developed expectations about how mathematics classes will run and how mathematics teachers will behave. Instructors of courses that differ from these expectations often find that students question their competence, the value of the course, or what they are expected to do in the class. Because Schoenfeld's course is an elective, if students decide he is not competent, that the course is not of value, or they don't understand what they will be asked to do, they may well leave the course. The students who stay in the course will need to understand what they are expected to do. Schoenfeld's path to achieving a "classroom mathematical community" includes the short-term goals of:

- establishing his "credentials";
- showing the students that heuristics are an important part of mathematics;
- giving the students a sense of what the course is about;
- communicating his expectations for classroom behavior.[17]

[17]Or as Schoenfeld put it (audiotaped discussion, May 22, 1991) "letting them know what they're in for."

The first two goals are related to an ancient pedagogical problem (cf. Plato's *Protagoras*): How can a student ignorant of a subject judge whether or not someone is capable of teaching it? The last two goals address a similar problem: How can students be asked to do mathematics (in some ways) like mathematicians if they have no idea how mathematicians do mathematics?

Schoenfeld's solution includes illustration and enactment. Here we use "enactment" in a somewhat theatrical sense. His introduction to the course and his treatment of the telescoping series portrayed him, though by different means, as both a member and critic of the mathematical community. The introduction was a monologue not involving the students. In contrast the caricature of the "typical calculus professor" enacted the distinction between how mathematics is presented in classrooms and how it is done by mathematicians using heuristics. Rather than telling the students about the drawbacks of traditional mathematics teaching, Schoenfeld depicted them in his caricature, then modeled the solution of a mathematician. During the discussion of the next two problems the students responded with traditional behaviors and then, with Schoenfeld's prompting, enacted some of the mathematical behaviors that he was trying to establish.

The discussion of the inscribed square illustrated the power of heuristics and the skill required to use them successfully. Students will, on average, not succeed in showing that a square can be inscribed in an arbitrary triangle, whether or not the heuristic Try an Easier Related Problem is suggested, so they will consider it a difficult problem. Because that difficult problem will yield, when an appropriate heuristic is suggested and its use scaffolded (Collins, Brown, & Newman, 1989), the inscribed square problem serves to show the power of heuristics in obtaining a solution—as well as the skill required to use them successfully. Allowing students to struggle may be an essential part of this process, both in showing the power of heuristics and Schoenfeld's ability to teach them. Students often aren't conscious of the important role that non-traditional teachers' suggestions and questions play in their progress toward a solution and sometimes conclude that such teachers don't know very much mathematics—otherwise they would tell them the answer. The first student presentation gave Schoenfeld an opportunity to mention a traditional student behavior, looking to the teacher for approval, and to have one of his expectations for classroom behavior enacted, that students not look to him for approval. This was a step toward the long-term goal of shifting the locus of authority away from the teacher and having the class, aided by Schoenfeld's "nasty questions," develop its own standards of correctness. The presentation also allowed the enactment of another expectation, that of "not just beautiful finished solutions but also explanations of how and why."

The discussion of the magic square served different goals: illustrating uses of heuristics and the theme of multiple solutions, and as a vehicle to engage students in discussion. It showed that different heuristics can be used singly or in combination to solve the same problem. (In contrast, the inscribed square showed that the same heuristic can be used in different ways on the same problem, a different illustration of the theme of multiple solutions.) Because it is easy to solve, the magic square could not be used to illustrate the power of heuristics in obtaining a solution. Instead it allowed students to focus on the use of heuristics. Students find generalizations of the magic square easy,[18] hence Schoenfeld could and did use it to have students enact another long-term goal of the course: That students take problems and make them their own by extension or generalization.

Pedagogy

Planning, direction, and authority

The sequence of problems and associated activities (Schoenfeld's introduction to the course, lectures, reflective presentations, student presentations, small-group work, and whole-class discussions) give an overall structure for the first days of the course. The problems are not chosen to cover content in the traditional sense, but to make certain points about heuristics and the course. Schoenfeld has used the telescoping series, inscribed square, and magic square for years. He was thus familiar with probable student responses to each in the contexts that he provides. For example, he knew what students are likely to do with the inscribed square without heuristics, and with the heuristics Try an Easier Related Problem and Relax a Condition. In this sense he controlled the class in the same way someone who digs a ditch controls the water flowing through it: The overall structure for the course channeled students' "natural" responses in directions that served many of Schoenfeld's goals. At some points (for example in the discussion of "easier related problems" for the inscribed square) without additional direction students might have become entangled in a fruitless exploration—an authentic mathematical experience, but one which was not likely to encourage students to

[18]"Easy" like "difficult" has an operational definition in this context—if, on average, students readily suggest generalizations of this problem, then generalizations of the problem can be said to be easy.

stay in the course or have confidence in its teacher. At such points Schoenfeld used his authority as a teacher and mathematician to guide the flow of the discussion. However, a goal of the course is that the teacher not remain the sole authority. Schoenfeld's delegation of authority to the students during Devon's presentation of his solution for the inscribed square problem was a step toward satisfying this long-term goal.

Opportunism

Within the structure imposed by the sequence of problems, the heuristics illustrated by the problems, and the activities surrounding them there is room for opportunism (Hayes-Roth & Hayes-Roth, 1979; Schoenfeld et al., 1992). Schoenfeld's knowledge of the problems (and of mathematics) allowed him to take advantage of student remarks such as Devon's suggestion about a solution for the magic square. Here the analogy might be to a navigator who knows how to get to a particular location in any event, but is able to take advantage of an unexpected wind not only to arrive, but arrive sooner. In this case, Devon's suggestion provided a context not only for discussing Working Backwards which Schoenfeld would do in any case (Schoenfeld, 1991), but showing it as a possible source for Devon's suggestion. This satisfied the goal of discussing Working Backwards and additional goals: incorporating student suggestions and again illustrating the notion of "not just beautiful finished solutions but also explanations of how and why."

Discourse and communication

Schoenfeld also used Pólya's method of questioning to involve the students in using heuristics. This method of questioning has social and cognitive aspects. On the one hand, Schoenfeld was asking for a response from the students which got them talking, helping to begin the community he wished to establish. On the other, the method of questioning scaffolded the students' applications of heuristics to particular cases. Other features of classroom communication (involved language, involved gestures, informal blackboard writings) suggested that the class was doing mathematics rather than being presented with mathematics.

In summary, we suggest that important elements in achieving the short-term goals for the first days of the course were:

- the sequence of problems Schoenfeld used;
- his knowledge of probable student responses to the problems;
- his knowledge of possible solutions to the problems and the heuristics that generate them;[19]
- his use and delegation of authority;
- his patterns of written and oral communication, and classroom discourse.

Implications

We will not venture to draw universal implications from a study of two days in one classroom. Nor is our intent to prescribe a teaching method. Instead, we hope this example will help to illuminate the difficult task of teaching mathematics.

Teaching is sometimes dichotomized as either transmission or discovery.[20] In the language of calculus reform (e.g., *UME Trends*, 1995) a teacher is either a "sage on the stage" or "guide on the side." Because it contains elements of both, Schoenfeld's teaching provides a counterexample to this notion. The form (though not always the content) of his presentations to the class contains traditional elements such as lecturing and blackboard expertise, but he combines these with non-traditional elements such as questioning and student work in groups. Furthermore, our analyses show that characterizing teaching in terms of use and frequency of methods such as lecture, small-group work, and whole-class discussion is inadequate because such characterizations omit the complex interaction between curriculum and pedagogy.

The problem solving course also counters the notion that a curriculum must be composed of individual strategies which are learned and practiced separately. Traditional algebra and calculus courses do just that—and instructors find to their dismay that students know the strategies, but may not know when to apply them. However, instructors who simply change course curricula without addressing student beliefs and expectations often find their students bewildered or resistant (see e.g., Cipra, 1995; Culotta, 1992). In turn, instructors often react by returning to traditional practices, and thus the status quo is maintained.

The curriculum and pedagogy of Schoenfeld's problem solving course suggests a way to alter unmathematical student habits—that they be enacted, mentioned, and

[19]What we have labeled here "knowledge of probable student responses to the problems" and "knowledge of possible solutions and the heuristics that generate them" is related to Shulman's notions of pedagogical content knowledge and subject matter knowledge (Fennema & Franke, 1992; Shulman, 1987).
[20]We thank Barbara Pence for reminding us of this.

revised. For example, the unsystematic listing of the triples in the magic square or the student looking to the teacher for approval were situations in which typical student behavior occurred (and was expected to occur), was commented upon, and an alternative enacted. Such situations can be engineered in other courses as they are in the problem solving course. Here is a brief sketch of an example: Students can be asked to work a problem that can be solved by a strategy that has just been taught, then asked to work a problem that is superficially similar but which can't be solved using the same strategy. Students' usual response is to try the most recently taught strategy. The instructor is then provided with an opportunity to mention that an important part of knowing a strategy is the recognition of the situations in which it can and can not be used—and to comment on the expectation that problems given in class are to be solved using the material that has been most recently taught. As with all curricular and pedagogical changes, this one would probably require several cycles of trial and refinement.

Final commentary

Pólya wrote in 1963:

> Everybody demands that the high school should impart to the students not only information in mathematics but know-how, independence, originality, creativity. Yet almost nobody asks these beautiful things for the mathematics teacher—is it not remarkable? . . . Here, in my opinion, is the worst gap in the subject matter knowledge of the high school teacher: he [or she] has no experience of active mathematical work. . . . (Pólya, 1981, p. 113)

Current reforms in precollege education make the experience of active mathematical work even more necessary for teachers now than in 1963. Moreover, studies suggest that prospective mathematicians, as well as prospective teachers, benefit from such an experience (Tucker, 1995). But it is still the case that few undergraduate courses offer students the opportunity to do, rather than ingest, mathematics (Tucker, 1995). Instructors have little opportunity to observe such courses and those who do may have little time in which to make sense of their curriculum and pedagogy. We hope in this article to have provided a useful substitute for a visit to one such course.

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Appendix A: Schoenfeld's introduction to the course

OK . . . Let me give you a little sense of what the course is about—a little bit of history. My name is Alan Schoenfeld. This is Math 67. It's a course in problem solving. It's a hands-on course. You'll spend most of the time in class solving problems, talking about them, doing mathematics. This is one of the rare courses in the department—in the country—where you actually *do* mathematics from the very beginning. The idea is to give you a chance to do some explorations, learn some neat stuff about problem solving.

Here's some of the history . . .

Way back in what seems like the dark ages—early 1970's—I was a young mathematician, had finished my degree, was a topologist and measure theorist happily proving theorems as beginning assistant professor, when I tripped across a book called *How to Solve It* written by George Pólya in 1947. Pólya is—was one of the eminent mathematicians of this century—probably one of the ten, fifteen greatest mathematicians of the 20th century. When he was about sixty, he decided—that was in the late 1940's—he decided: "You know, I've led a long and productive life. It's time for me to sit down and maybe think about start writing some things that would help other people to do and learn about mathematics." So he wrote this little thing called *How to Solve It*, in which he did a lot of introspection, said basically: "You know, there are some things that seem to be productive ways to solve problems, for me, for other mathematicians—ways of thinking that we've picked up, tricks of the trade that enable us to be really successful at solving problems. And they help us a lot. Maybe if I wrote them down—shared them—it would make life a little bit easier for other people as well." He went on, stayed in that business for another thirty or so years, productively until his mid-nineties.

I read the book in 1974 when I was a very young mathematician, and had a very funny reaction to it. I started out, read a few pages . . . and he said, "Mathematicians do this." I read a few more . . . he said, "Mathematicians do this." And I started to smile. "Hot damn, I must be a real mathematician—I do all the things Pólya says they do!" Then got pissed off, and said, "Hey, wait a sec. You know, here I am. I finished an undergraduate career. I went through an entire career as a graduate student. I'm a young professional. *Now* for the first time I'm reading about these tricks of the trade. Why didn't they tell me when I was a freshman, and save me the trouble of discovering all of them for myself? Maybe it's a version of the medieval trial by gauntlet: the only people we want are the ones who succeed without knowing the rules." I don't know . . .

So I asked around, and I asked some of the people who prepared people for problem solving competitions. There is a thing called the Putnam exam that a lot of people study for—if you do well on it you're guaranteed admission to the graduate school of your choice. I asked people who were in mathematics education. And the uniform response I got was:

Every mathematician I talked to said, "Yup, Pólya is absolutely right. My guts tell me he is right. I do the things he writes about." And every problem solving coach, and everyone I talked to who was involved in getting students to solve problems better, said, "You know, it's a strange thing. I've never been able to use Pólya's ideas in such a way that my students actually wound up being better at it. So I don't use them very much anymore."

That was the intellectual dilemma that in the mid-1970's got me to turn to problem solving and got me to focus on it, as the main thing that I would do for the next 15 years. 'Cause on the one hand, I believe in the ideas that Pólya had. And on the other hand I believe the people who said, "As constituted, Pólya's ideas don't work." So what I've been involved in largely for the past 15 years is figuring out how to make those ideas work—figuring out what it is that it takes to use the kind of problem solving strategies that he talks about, effectively; and through the years building and changing and modifying this course so that it works. And the one thing that I can guarantee you is: It does work. By the end of this course you will have an arsenal of problem solving tools and techniques that will enable you to be much more successful, not only in solving problems that you've been shown how to solve, but also at encountering new things and making sense of them—which is something that your math courses don't normally train you how to do.

I'll tell you about the ideal goals for a course like this and then, what I actually did as evidence of what you can expect to be in for; and then I'll tell what the structure of the course will be; and then I'll stop talking and we'll do what we should do, which is get on to solving problems.

The goal of this course is to give you enough experience and exposure to solving problems and learning about the tools and techniques of the trade so that you walk out of this course a far more resourceful and better problem solver, . . . again, at not only at dealing with the kinds of things I've shown you to deal with in the course, but also when you encounter something new—having at your disposal a set of techniques that will enable you to make progress on and make sense of a problem that you haven't been shown how to solve.

Here's the ideal test for the course. I've been in problem solving for fifteen, twenty years. There are other people who have massive reputations for such things. There's a guy named Paul Halmos who used to be editor of the *American Mathematical Monthly*, who's been writing about problem solving forever. The kind of thing you might want to do is say to Halmos, "Hey, look. Schoenfeld's gonna teach his problem solving course. Here are the backgrounds of his students. Here are the kind of people you can expect to see in the course. What we'd like you to do is make up two tests. Make up a matched pretest and posttest, which in some sense are identical in content. And, he won't know what's in your tests; you won't know what's in his course. If he really does what he says he does, then his students should do far better on the posttest than they did on the pretest. And other kids taking, say math H50A, or analysis, or Riemann surfaces, shouldn't really show any performance difference." *That* would be the sort of iron-clad test that I did something in this course.

I never had the nerve to do that [giggles from the class, Schoenfeld smiles]. But I'll tell you what I did—which was worse—some two or three versions of the course ago. I gave an in-class final exam. (Now I actually prefer, although rules require in-class exams . . . what you'll be doing is a couple of take-homes, for the mid-term and final, just a *pro forma* simple in-class written exam.)

I gave an in-class final, and there were three parts to the final exam. The first part was problems like the problems we solved in class. No surprise, you expect people to do well on those. The second part was problems that could be solved by the methods that we used in class—but ones for which if you looked at them you couldn't recognize that they had obvious features similar to the ones that we'd studied in class. So yes, you had the tools and techniques, but you had to be pretty clever about recognizing that they were appropriate. And, the class did pretty well on those too. Part three of the final exam . . . There's a collection of books called the *Hungarian Problem Books* which have some of the nastiest mathematical problems known to man and woman. I went through those, and as soon as I found a problem I couldn't make any sense of, whatsoever, I put it on the final. (I know that makes you feel good.) [Laughter from class, Schoenfeld smiles.] The class did spectacularly well, and actually wound up solving some problems I didn't, OK—which is pretty good proof that amazing things happen in the class. And they happen, I think, because we're serious about really doing mathematics—which is the name of the game. So let me tell you a bit about what's gonna happen.

Most days I'm going to walk in (today being only a slight exception in the sequence), and hand out a bunch of problems. I've got enough here to probably keep us busy for two days or so. And what you're seeing here is unusual, because you won't be seated in rows watching me talk. Instead you're going to break into groups of three or four or five, and work on problems together. As you're working on them, I'll circulate through the room, occasionally make comments about the kinds of things you're doing, respond to questions from you. But, by and large, I'll just nudge you to keep working on the problems.

Then at some point I'll call us to order as a group, and we'll start discussing the things that you've done, and talk about the things that you've pushed and why; what's been successful, what hasn't. I'll mention a variety of specific mathematical techniques as we go through the problems. Many of the problems are chosen so that they illustrate useful techniques. So you'll work on one for a while; may or may not make some progress; and then we'll talk about it. And as we talk about it what I'll do is indicate some of the problem solving strategies that I know, and that are in the literature, that might help you make progress on this problem, and progress on other problems. And we'll use those strategies as a means of bootstrapping our way into the problems.

The course is pretty wide-open. I've taught it now seven or eight times, every other year, thereabouts. And every year the course is different, because it turns out to be a creature of the people in it. Everything mathematical is fair game in here, which means that you'll find if we get turned on by a problem, we'll push it. If we see interesting things, we'll pursue that particular domain of mathematics for a while. The bottom line is: I'm happy when we're doing real mathematics. What that means may not be clear to you now but it will become increasingly clear during the semester . . . and this for me is the course I most love teaching 'cause it's the greatest fun and the one that is most involving for both me and my students.

As I said, what's going to happen is: Most days I'll hand out problems. We'll work on them in class. Some of the problems can be solved fairly fast and some of them merely serve as introductions to more conjectures and more problems. Other problems may be things that we visit for two or three days—of classes—maybe even a week or two as we do something, find something interesting in it, but don't make enough progress as a group; so I'll say, "Fine, let's get back to it next time" and we'll keep working on the problem over a period of days. So what we do the vast majority of times in here is just do and talk mathematics. And learn some mathematics.

The grading for the course. Well, a week or two into the class I'll give you the opportunity to write out a problem or two for me so that I can get a sense of the kind of writing you do, and give you some feedback on the kind of writing I expect. The first main thing we do is: about half-way through the course I'll give you a two-week take-home. It'll consist of about ten problems and they will occupy you for a long time. But you'll make progress on them and you'll do reasonably well on them. And then, the final. Again, the department formally requires me to give an in-class final, so I usually wind up giving a one-problem in-class final to meet the rules and regulations. That's about ten percent of the final exam grade. The rest of it is another take-home that you'll have two weeks to work on. There are some funny rules, which are that:

What counts is not simply the answer, what counts is doing mathematics. And that means, among other things, if you can find two different ways to solve a problem, you'll get twice as much credit for it. If you can extend the problem and generalize it and make it your own, you'll get even more. The bottom line is, I'd like to have you doing some mathematics and I will do everything I can—including using grading—as a device for having you do that. Grades turn out to be pretty much of a non-issue in the course. What usually happens is, people get sucked into it. You get out of the course what you put into it, basically—that becomes clear. If you haven't done much through the semester, you'll find you're not ready to do terribly well on the midterm and final; if you have, you'll find that you'll do fairly well. Anyone who kicks in and just participates actively during the semester (it's obvious) and no one who's done that has ever gotten less than C+. Typically, the grades have been mostly As and Bs because people have done very well on what's demonstrably good mathematics. So we can say more about grading when the time comes, but it really will turn out to be a non-issue. Today what I'm going to do, is just now, stop talking—that will make us all feel better—and then hand out a bunch of problems. They're especially chosen for the first day, to make a couple of points—rhetorical points about problem solving strategies. I will make those points clearer after you've worked on the problems for a while. But I think designed to give you a sense of what the rest of the semester is going to be like. OK. Anybody got any questions? [Pauses, no questions.]

OK. Then what you ought to do is break into groups of three or four or thereabouts; and you should be prepared to work together . . .

[Refers to the fact that someone is videotaping the class.] Oh, the man with the camera. As I said, this is—this has been part of my own enterprise now for about 15 years. And over the years the course has developed and grown in interesting ways. And one of the things that I like to do is make sense of what happens in the course myself. I often write about it as part of my research, as well as part of my teaching. (They go hand in hand because my research is about understanding the nature of mathematical thinking and using that understanding to help build courses like this.) So that camera is a record for me of what's happened in the course, so that I can reflect on it, in the hope of making sense of it and making it better for the next round of students.