Computing with sheaves and sheaf cohomology in algebraic geometry: preliminary version

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1 Introduction

These notes are still in a preliminary form. They are being developed for lectures I am giving at the Arizona Winter School in Tucson, March 11-15, 2006. Any suggestions or corrections would be greatly appreciated!

Cohomology of sheaves has become a tremendously useful method in algebraic geometry. Often the way that cohomology is introduced is quite abstract, and although it is possible to compute it explicitly in simple cases with the properties that are apparent from its definiton, it can be hard to obtain detailed information, or to compute these groups for more involved examples.

The purpose of these lectures is twofold: (1) to present computational methods which allow us to determine the cohomology of coherent sheaves on projective varieties (or schemes), and (2) to present many examples and applications of their use.

Throughout these notes, $S = k[x_0, \ldots, x_n]$ will denote the homogeneous coordinate ring of $\mathbb{P}^n$ over a field k, k not necessarily algebraically closed. If $X = V(I) \subset \mathbb{P}^n$, we will denote by $R = S/I$ its homogeneous coordinate ring.

If $\mathcal{F}$ is a coherent sheaf on $X = V(I)$, what do we mean when we say that we wish to compute the cohomology of $\mathcal{F}$? We generally mean one the following:

- The k-vector space $H^i(X, \mathcal{F})$, or its dimension $h^i(X, \mathcal{F})$, or
- The R-module
$$H^i_*(X, \mathcal{F}) = \bigoplus_{d \in \mathbb{Z}} H^i(X, \mathcal{F}(d)).$$
Since it turns out that this is sometimes not finitely generated, we might also wish to find, for an $e \in \mathbb{Z}$,
$$H^i_{\geq e}(X, \mathcal{F}) = \bigoplus_{d \geq e} H^i(X, \mathcal{F}(d)).$$

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As long as you are willing to accept our definitions of cohomology and coherent sheaf, you will not need to know much more about these concepts. In this case, hopefully these lectures will convince you of their importance, and make it easier to study the more abstract theory. If you are already familiar with coherent sheaves and their cohomology, hopefully there will be some techniques presented here that you will find useful!

Many of the methods presented here were done by Serre [12] in the 1950's. David Eisenbud has a nice chapter in Vasconcelos' book [14] on some methods of computing cohomology of coherent sheaves. Greg Smith's paper on computing global Ext [**?**] is very clearly written, and is also useful for computing sheaf cohomology (since global Ext is a simple generalization of sheaf cohomology) and of course global Ext. It also has *Macaulay2* code (which has since been incorporated in *Macaulay2* [5]) and several explicit examples. Many of the techniques presented here for projective space also work over products of projective spaces and over toric varieties (see Eisenbud, Mustata and Stillman [4]). The explicit Bernstein-Gelfand-Gelfand correspondence and using it to compute sheaf cohomology comes from Eisenbud-Floystad-Schreyer [3]. See also [1] and [13]. The technique also generalizes to allow one to compute higher direct images of coherent sheaves under projective morphisms. See Eisenbud, Hulek and Schreyer [**?**].

2 A brief introduction to sheaf cohomology

One way to use sheaf cohomology is as a black box, using it via its properties, such as the long exact sequence, and its values on specific sheaves. In fact, these properties provide an axiomatic definition. In many cases, this works quite well. Our plan in this section is to first define sheaf cohomology using the Cech complex, present some basic properties of cohomology, do a simple example by hand, and do a second example showing how to compute cohomology with *Macaulay2*.

Our setting: Let $\mathcal{F}$ be a coherent sheaf on $X \subset \mathbb{P}^n$. Let $\{U_i \mid 0 \le i \le m\}$ be an open affine cover of X. The standard open affine cover consists of the affine open sets $U_i = X \setminus V(x_i) \subset \mathbb{P}^n$ in $X \subset \mathbb{P}^n$. For a subset $\lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_p\}$, let $|\lambda| = p$, and $U_\lambda := \bigcap_{i=0}^{p} U_{\lambda_i}$.

Definition 2.1 (Cech complex). *For $0 \le p \le m$, Let $\mathcal{C}^p = \bigoplus_{|\lambda|=p} \mathcal{F}(U_\lambda)$, and let*

$$\sigma_p : \mathcal{C}^p(\mathcal{F}) \longrightarrow \mathcal{C}^{p+1}(\mathcal{F})$$

be the natural map

$$\sigma_p : (f_{i_0 i_1 \ldots i_p}) \mapsto (g_{j_0 \ldots j_{p+1}}),$$

where

$$g_{j_0 \ldots j_{p+1}} = \sum_{i=0}^{p+1} (-1)^i f_{j_0 \ldots \hat{j_i} \ldots j_{p+1}}.$$

The Cech complex $\mathcal{C}(\mathcal{F})$ *of* $\mathcal{F}$ *is the complex*

$$0 \longrightarrow \mathcal{C}^0(\mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{F}) \longrightarrow \ldots \longrightarrow \mathcal{C}^m(\mathcal{F}) \longrightarrow 0.$$

Let $H^i(\mathcal{F}) = H^i(X, \mathcal{F})$ *be the* i*-th cohomology* $H^i(\mathcal{C})$ *of this complex (of infinite dimensional* k*-vector spaces). It is possible to check that choosing another affine open cover would result in isomorphic cohomology groups.*

With some work (which Serre did in the 1950's (see [12]), and which we will be in a position to prove later), one can prove the following facts:

Proposition 2.2 (Facts about cohomology)**.** *If* $\mathcal{F}$ *is a coherent sheaf on* $\mathbb{P}^n$*, then*

(a) $H^i(\mathcal{F})$ *is a finite dimensional* k*-vector space, for all* $0 \le i \le n$*.*
(b) If $d = \dim \operatorname{supp} \mathcal{F}$*, then for* $i > d$*,* $H^i(\mathcal{F}) = 0$*.*
(c) For $d \gg 0$*, and* $i > 0$*,* $H^i(\mathcal{F}(d)) = 0$*.*
(d) [Long exact sequence] If

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is a short exact sequence of coherent sheaves on $\mathbb{P}^n$*, then there are connecting homomorphisms* $H^i(\mathcal{F}'') \longrightarrow H^{i+1}(\mathcal{F}')$ *such that the resulting sequence*

$$0 \longrightarrow H^0(\mathcal{F}') \longrightarrow H^0(\mathcal{F}) \longrightarrow H^0(\mathcal{F}'') \longrightarrow H^1(\mathcal{F}') \longrightarrow \ldots$$

is exact.

It is not too hard from the definition to find the cohomology of all degree twists of the structure sheaf of $\mathbb{P}^n$.

Proposition 2.3. *(Serre [12]) For any integer* d*,*

$$\begin{aligned} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) &= S_d \\ H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) &= 0 \\ H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) &= S'_{-n-1-d}, \end{aligned}$$

where $S = k[x_0, \ldots, x_n]$*, and the prime (') denotes* k*-vector space dual.*

Example 2.4. As a simple example, if $X = V(f) \subset \mathbb{P}^3$ is a cubic surface, we can compute the cohomology of $\mathcal{O}_X(d)$, for all d. Write $\mathcal{O}$ for $\mathcal{O}_{\mathbb{P}^3}$, and consider the following exact sequence of coherent sheaves on $\mathbb{P}^3$:

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{f} \mathcal{O} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Tensoring with a locally free sheaf (such as $\mathcal{O}(d)$) leaves this sequence exact, so

$$0 \longrightarrow \mathcal{O}(d-3) \longrightarrow \mathcal{O}(d) \longrightarrow \mathcal{O}_X(d) \longrightarrow 0$$

is still exact. The long exact sequence in cohomology, combined with the previous proposition gives: $0 \longrightarrow S_{d-3} \longrightarrow S_d \longrightarrow H^0(\mathcal{O}_X(d)) \longrightarrow 0$ is exact, and $H^1(\mathcal{O}_X(d)) = 0$, for all d, and $H^2(\mathcal{O}_X(d))$ is the cokernel of the map $S'_{-1-d} \longrightarrow S'_{-4-d}$. So $H^2(\mathcal{O}_X(d)) = 0$, for $d \ge 0$.

This technique was called “eye-balling” by David Eisenbud, in his chapter on computing cohomology ([14]). With enough exact sequences, and other information about the sheaves, this method can be quite powerful, at least once one has some idea of what these cohomology modules represent.

So, what are the problems with this method?

(1) It is hard to reconstruct the module structure.

(2) If not enough of the cohomology spaces are zero, it is hard to deduce the dimensions of specific cohomology groups.

Example 2.5 (Sheaf cohomology with *Macaulay2*: The Fermat cubic surface). Let’s consider a simple example of using sheaf cohomology as a black box in *Macaulay2*. Consider the Fermat cubic surface in $\mathbb{P}^3$.

```
i1 : S = QQ[a..d];

i2 : R = S/(a^3+b^3+c^3+d^3);
```

The projective variety corresponding to this ring is

```
i3 : X = Proj R

o3 = X

o3 : ProjectiveVariety
```

The structure sheaf of X:

```
i4 : OO_X

o4 = OO
       X

o4 : SheafOfRings
```

Twists of sheaves are constructed using standard notation.

```
i5 : OO_X(4)

        1
o5 = OO  (4)
       X

o5 : coherent sheaf on X, free
```

The cohomology operator is `HH`.

```
i6 : HH^0(OO_X)

       1
o6 = QQ

o6 : QQ-module, free

i7 : (rank HH^2(OO_X), rank HH^2(OO_X(-1)), rank HH^2(OO_X(-2)))

o7 = (0, 1, 4)

o7 : Sequence

i8 : rank HH^1(OO_X(-3))

o8 = 0
```

The module structure for a truncation of $H^i_*(\mathcal{F})$ is done using the following syntax. In this case we are finding the truncation at degree -2.

```
i9 : HH^0(OO_X(>=-2))

      1
o9 = R

o9 : R-module, free
```

The sheaf associated to a graded module:

```
i10 : M = ideal(a^3+b^3)

                3    3
o10 = ideal(- c  - d )

o10 : Ideal of R

i11 : F = sheaf M

o11 = image | -c3-d3 |

                                            1
o11 : coherent sheaf on X, subsheaf of OO
                                           X

i12 : HH^0(F(3))

        1
o12 = QQ

o12 : QQ-module, free
```

We haven't discussed the cotangent sheaf yet, but it may be obtained as follows.

```
i13 : cotangentSheaf X

o13 = cokernel {2} | 0  0  d  c  0   a2  b2 0  |
               {2} | 0  d  0  -b a2  0   c2 0  |
               {2} | 0  -c -b 0  0   0   d2 a2 |
               {2} | d  0  0  a  b2  -c2 0  0  |
               {2} | -c 0  a  0  0   -d2 0  b2 |
               {2} | b  a  0  0  -d2 0   0  c2 |

                                            6
o13 : coherent sheaf on X, quotient of OO  (-2)
                                           X

i14 : HH^0(cotangentSheaf X)

o14 = 0

o14 : QQ-module
```

If the sheaf $\mathcal{F}$ is defined on a projective subvariety or subscheme $X \subset \mathbb{P}^n$, then $\mathcal{F}$ can be thought of as a sheaf $i_*\mathcal{F}$ on $\mathbb{P}^n$, where $i : X \to \mathbb{P}^n$ is the inclusion map. It is easy to check that $H^i(X, \mathcal{F}) \cong H^i(\mathbb{P}^n, i_*\mathcal{F})$. Consequently, throughout these notes, we may as well assume that our sheaves are defined on $\mathbb{P}^n$, and we will write $H^i(\mathcal{F})$ for the cohomology group $H^i(X, \mathcal{F}) = H^i(\mathbb{P}^n, i_*\mathcal{F})$.

3 Background

We review some basic concepts and techniques from algebra, such as free resolutions, Koszul complexes, and Ext modules. We use this as an opportunity to show how to compute with these objects in *Macaulay2*.

An interesting example is the rational quartic curve $X \subset \mathbb{P}^3$. This is the image of the polynomial map $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ where $(s,t) \mapsto (s^4, s^3t, st^3, t^4)$.

```
i15 : kk = ZZ/32003;

i16 : ringP1 = kk[s,t];

i17 : S = kk[a..d];
```

The ideal of the rational quartic is the kernel of the following ring map.

```
i18 : F = map(ringP1,S,{s^4,s^3*t,s*t^3,t^4})

                         4   3      3   4
o18 = map(ringP1,S,{s , s t, s*t , t })

o18 : RingMap ringP1 <--- S

i19 : I = kernel F

                               3      2     2    2    3    2
o19 = ideal (b*c - a*d, c  - b*d , a*c  - b d, b  - a c)

o19 : Ideal of S

i20 : R = S/I

o20 = R

o20 : QuotientRing
```

As expected, I has codimension 2, degree 4, and genus 0.

```
i21 : (codim I, degree I, genus I)

o21 = (2, 4, 0)

o21 : Sequence
```

As far as *Macaulay2* is concerned, R is a quotient ring, a very different object from the S*-module* S/I. We may form the S-module as follows.

```
i22 : M = S^1/I

o22 = cokernel | bc-ad c3-bd2 ac2-b2d b3-a2c |

                                   1
o22 : S-module, quotient of S
```

3.1 Modules

We will mainly be interested in graded modules over R or S. First, let's check that the S-module $M = S/I$ defined above is graded.

```
i23 : isHomogeneous M

o23 = true
```

Recall that the d-th twist of $M = \bigoplus_{d\in\mathbb{Z}} M_d$ is the graded module $M(d)$ which is the same module M, with a new grading: $M(d)_e := M_{d+e}$.

```
i24 : S^{3}

       1
o24 = S

o24 : S-module, free, degrees {-3}
```

Note that -3 is the degree of the generator of $S(3)$.

In *Macaulay2*, to obtain the module $M(3)$, tensor the module M with the free module $S(3)$. (`**` is used as the tensor product operator)

```
i25 : M ** S^{3}

o25 = cokernel {-3} | bc-ad c3-bd2 ac2-b2d b3-a2c |

                              1
o25 : S-module, quotient of S
```

The *truncation* $M_{\geq e}$ of a graded module $M = \bigoplus_{d \in \mathbb{Z}} M_d$ is defined to be the graded module

$$M_{\geq e} := \bigoplus_{d \geq e} M_d.$$

```
i26 : truncM = truncate(1,M)

o26 = subquotient (| a b c d |, | bc-ad c3-bd2 ac2-b2d b3-a2c |)

                                 1
o26 : S-module, subquotient of S
```

Notice that the result is a subquotient module: it is generated by the image of the first matrix, modulo the image of the second matrix. We can obtain a (graded) module isomorphic to this which is a quotient of a free module by using `prune`.

```
i27 : prune truncM

o27 = cokernel {1} | 0  a  0  b  c  0  0  0   0   0   |
               {1} | b  0  0  0  -d a  0  c2  0   0   |
               {1} | 0  0  c  -d 0  0  a  -d2 -bd b2  |
               {1} | -d -d -d 0  0  -c -b 0   c2  -ac |

                              4
o27 : S-module, quotient of S
```

Each row corresponds to a generator of the module, in this case there are four generators, $e_1, \ldots, e_4$, each having degree 1. Each column corresponds to a relation on these generators. For example, the first column corresponds to the relation $be_2 - de_4 = 0$.

Finally, we will be interested in vector spaces M_d of graded modules M. Use `basis(d,M)` to get a map to M whose image is a k-basis of the degree d part:

```
i28 : basis(2,M)

o28 = | a2 ab ac ad b2 bd c2 cd d2 |

o28 : Matrix
```

If all you want is the dimension, you may use

```
i29 : numgens source basis(2,M)

o29 = 9
```

or

```
i30 : hilbertFunction(2,M)

o30 = 9
```

3.2 Free resolutions

Let M be a graded R-module. An exact sequence of free R-modules and maps of degree 0 of the form

$$\cdots \to F_r \to F_{r-1} \to \ldots F_1 \to F_0 \to M \to 0$$

is called a *graded free resolution* of M.
The Hilbert Syzygy theorem shows that if $R = k[x_0, \ldots, x_n]$ then any finitely generated module admits a free resolution with at most $n+1$ free modules.
Among all free resolutions of M there is a *minimal* one $\mathbb{F}$ with the property that every other resolution $\mathbb{L}$ of M is of the form $\mathbb{L} = \mathbb{F} \oplus \mathbb{T}$ where $\mathbb{T}$ is a sum of trivial exact $0 \to R(d) \to R(d) \to 0$ complexes. The *minimal free resolution* $\mathbb{F}$ can be recognized by the fact that there are no nonzero entries of degree zero in any of its matrices and allows us to define several invariants of a graded module M. If $\mathbb{F}$

$$\mathbb{F} : \cdots \to F_r \to \cdots \to F_1 \to F_0 \to M \to 0$$

is the minimal free resolution of M with

$$F_i = \bigoplus_{j=0}^{b_i(M)} R(-a_{i,j}(M))$$

we define:

- Projective dimension: $pdim(M) \ = \ \max\{r \mid F_r \neq 0\}$
- Total Betti numbers: $b_i(M)$
- Graded Betti numbers: $b_{i,d}(M) := |\{j \mid a_{i,j}(M) = d\}|$
- (Castelnuovo-Mumford) regularity:

$$reg(M) \ = \ \max\{a_{i,j}(M) - i \mid 0 \leq i, \ 1 \leq j \leq b_i(M)\}$$

By writing these values in terms of $\mathrm{Tor}_i^R(M,k)$, it is easy to see that these are independent of the given minimal free resolution.

Note that over an arbitrary ring R (for example $R = S/I$ for a homogeneous ideal I) a module M will rarely admit a finite free resolution (even if it is graded). In this case, $pdim_R(M) = \infty$ and $reg(M)$ may be finite or infinite.

Castelnuovo-Mumford regularity is often defined in terms of cohomology. The relationship with this definition will become clear after the local duality theorem.

Constructing a free resolution efficiently is a simple generalization of Buchberger's algorithm for computing Groebner bases. For details on how to perform the computation, and further references, see [9].

Let's illustrate these concepts with the aid of Macaulay2 using the rational quartic curve defined earlier.

```
i31 : M

o31 = cokernel | bc-ad c3-bd2 ac2-b2d b3-a2c |

                             1
o31 : S-module, quotient of S

i32 : C = res M

       1      4      4      1
o32 = S  <-- S  <-- S  <-- S  <-- 0

      0      1      2      3      4

o32 : ChainComplex

i33 : C.dd

           1                                     4
o33 = 0 : S  <---------------------------------- S  : 1
                | bc-ad b3-a2c ac2-b2d c3-bd2 |

           4                             4
      1 : S  <-------------------------- S  : 2
                {2} | -b2 -ac -bd -c2 |
                {3} | c   d   0   0   |
                {3} | a   b   -c  -d  |
                {3} | 0   0   a   b   |

           4                 1
      2 : S  <------------- S  : 3
                {4} | d  |
                {4} | -c |
                {4} | -b |
                {4} | a  |

           1
      3 : S  <----- 0 : 4
                0

o33 : ChainComplexMap
```

Each map is obtained using indexing:

```
i34 : C.dd_2

o34 = {2} | -b2 -ac -bd -c2 |
      {3} | c   d   0   0   |
      {3} | a   b   -c  -d  |
      {3} | 0   0   a   b   |

                4      4
o34 : Matrix S  <--- S
```

Each free module `C_0, C_1, ...`, in the resolution is a graded free module. One may view the degrees (i.e. negative twists) as follows.

```
i35 : degrees C_1

o35 = {{2}, {3}, {3}, {3}}

o35 : List
```

These are the degrees of the generators of each summand. Therefore,

$$C_1 \;=\; S(-2) \oplus S(-3)^3$$

```
i36 : degrees C_2

o36 = {{4}, {4}, {4}, {4}}

o36 : List
```

The regularity and projective dimension of M:

```
i37 : regularity M

o37 = 2

i38 : pdim M

o38 = 3
```

The projective dimension is the length of the minimal resolution C.

```
i39 : length C

o39 = 3
```

A useful way to see the graded Betti numbers is the `betti` command.

```
i40 : betti C

o40 = total: 1 4 4 1
          0: 1 . . .
          1: . 1 . .
          2: . 3 4 1
```

This says that there is one generator of the ideal in degree 2, 3 in degree 3. There are four first syzygies, all in degree 4, and one second syzygy in degree 5. Observe that the regularity is the index of the last non-zero row in the betti diagram, while the projective dimension is the index of the last nonzero column (in this case 3).

For some methods of computing cohomology, we will require knowledge about the degrees that occur in the resolution. The maximum or minimum degree at the i spot can be obtained via:

```
i41 : first max degrees C_1

o41 = 3

i42 : first min degrees C_1

o42 = 2

i43 : degreeRange = (M, i) -> (
           C := res M;
           d := apply(degrees C_i, first);
           (min d, max d));
```

```
i44 : degreeRange(M,1)

o44 = (2, 3)

o44 : Sequence
```

It is possible for the computation of the free resolution of $M = F/J$ to be too demanding in terms of either time or computer memory. In these cases, knowing the free resolution of $F/in(J)$, where $in(J)$ is the submodule of initial (lead) terms of J will also provide us with bounds. More precisely:

Proposition 3.1. *Let $J \subset F$ be a submodule of a free S-module F, set $M = F/J$. Fix a monomial order $>$ on F, and let $in(J) \subset F$ be the submodule generated by the lead monomials of a Groebner basis of J, under the order $>$. Then for every $i \geq 0$ and every degree d, the graded Betti numbers satisfy*

$$b_{i,d}(M) \leq b_{i,d}(F/in(J)).$$

These bounds however, are generally not sharp. As an example, consider the normal bundle of the rational quartic in $\mathbb{P}^3$.

```
i45 : N = prune Hom(I/I^2,S^1/I);

i46 : J = presentation N;

               7       15
o46 : Matrix S  <--- S

i47 : betti res coker J

o47 = total: 7 15 9 1
         -1: 7 12 5 .
          0: .  3 4 1

i48 : inJ = leadTerm gens gb J;

               7       21
o48 : Matrix S  <--- S

i49 : betti res coker inJ

o49 = total: 7 21 19 5
         -1: 7 12  9 2
          0: .  7  8 3
          1: .  2  2 .
```

3.3 The Koszul complex

Let $W = S^{n+1}$, with basis $e_0, \ldots, e_n$. The exterior p-th power $\wedge^i(W)$ is the free S-module with basis

$$\{e_I = e_{i_1} \wedge \ldots e_{i_p} \mid I = \{0 \leq i_1 < \cdots < i_p \leq n\}\}.$$

There are several equivalent ways to define the Koszul complex. Here is one explicit version:

Definition 3.2 (Koszul complex). *Let $f_0, \ldots, f_n \in S$. The Koszul complex $K(f_0, \ldots, f_n)$ is the complex*

$$0 \longrightarrow S \xrightarrow{d_{n+1}} \wedge^n W \xrightarrow{d_n} \cdots \longrightarrow \wedge^2 W \xrightarrow{d_2} W \xrightarrow{d_1} S \longrightarrow 0\ ,$$

where

$$d_p(e_I) := \sum_{j=1}^{p} (-1)^{p+1} f_j e_{I \setminus i_j}.$$

One easily checks that this is a complex.

Proposition 3.3. *If $\{f_0, \ldots, f_n\} = \{x_0, \ldots, x_n\}$ then the Koszul complex is a minimal free resolution of $k = S/(x_0, \ldots x_n)$:*

$$0 \longrightarrow S(-n-1) \xrightarrow{d_{n+1}} \wedge^n W(-n) \xrightarrow{d_n} \cdots \longrightarrow \wedge^2 W(-2) \xrightarrow{d_2} W(-1) \xrightarrow{d_1} S \longrightarrow k \longrightarrow$$

For a proof, see almost any book on commutative algebra. In particular, Matsumura [**?**] has a very nice presentation.

Let's see the Koszul complex explicitly in the case of three elements:

```
i50 : A = kk[a,b,c];

i51 : koszul(1,vars A)

o51 = | a b c |

               1       3
o51 : Matrix A  <--- A

i52 : koszul(2,vars A)

o52 = {1} | -b -c 0  |
      {1} | a  0  -c |
      {1} | 0  a  b  |

               3       3
o52 : Matrix A  <--- A

i53 : koszul(3,vars A)

o53 = {2} | c  |
      {2} | -b |
      {2} | a  |

               3       1
o53 : Matrix A  <--- A
```

The Koszul complex is a minimal free resolution of k.

```
i54 : C = res coker vars A

       1      3      3      1
o54 = A  <-- A  <-- A  <-- A  <-- 0

      0      1      2      3      4

o54 : ChainComplex
```

```
i55 : C.dd

             1                 3
o55 = 0 : A  <------------- A  : 1
                | a b c |

             3                        3
      1 : A  <-------------------- A  : 2
                {1} | -b -c 0  |
                {1} | a  0  -c |
                {1} | 0  a  b  |

             3                  1
      2 : A  <-------------- A  : 3
                {2} | c  |
                {2} | -b |
                {2} | a  |

             1
      3 : A  <----- 0 : 4
                0

o55 : ChainComplexMap
```

3.4 Hom, Ext, and Tor modules

Hom, Ext, and Tor can all be computed efficiently, all we need is the knowledge of how to compute free resolutions and syzygies.

The key is the ability to compute the kernel of a homomorphism $\phi : M \longrightarrow N$ of S-modules. If M and N are (finitely generated) free S-modules, then the module of syzygies (i.e. the kernel) is a byproduct of computing the Groebner basis of the submodule $\mathrm{Im}(\phi)$ of M. If at least one of M and N is not free, it is a good exercise to see how, using Groebner bases and syzygies, one can determine the kernel of ϕ.

Exercise 3.4. *Suppose that R is a Noetherian ring such that one may compute a basis for the kernel of a map of free modules. Let M and N be finitely generated R-modules, and let*

$$M_1 \xrightarrow{m} M_0 \longrightarrow M \longrightarrow 0,$$

and

$$N_1 \xrightarrow{n} N_0 \longrightarrow N \longrightarrow 0$$

be presentations, where M_i and N_j are all free modules.

(a) Show how to compute a generating set for the kernel of a map $F \longrightarrow N$, where F is a finitely generated free R-module.

(b) A homomorphism $\phi : M \longrightarrow N$ corresponds (in a nonunique manner) to a matrix $f : M_0 \longrightarrow N_0$ such that $\mathrm{Im}(fm) \subset \mathrm{Im}(n)$. In terms of the three matrices m, n, and f, find the kernel of ϕ.

(c) Given a complex of R-modules

$$M' \xrightarrow{\beta} M \xrightarrow{\alpha} M'',$$

compute its homology $\frac{\ker \alpha}{\mathrm{Im}\, \beta}$.

3.4.1 Hom **modules**

One construction that we will use frequently is the module of homomorphisms. For example, the S-module associated to the normal bundle (or sheaf) of $X = V(I)$ in $\mathbb{P}^n$ is $\mathrm{Hom}_S(I, S/I) = \mathrm{Hom}_S(I/I^2, S/I)$.

If M and N are finitely generated R-modules, $\mathrm{Hom}_R(M, N)$ may be computed using syzygies in the following manner. For a free R-module $F = \bigoplus_i R(-a_i)$,

$$\mathrm{Hom}_R(\bigoplus_i R(-a_i), N) \;=\; \bigoplus_i \mathrm{Hom}_R(R, N(a_i)) \;=\; \bigoplus_i N(a_i) \;=\; F^* \otimes_R N,$$

and if $G \xrightarrow{\phi} F \longrightarrow M \longrightarrow 0$ is a presentation of M, then we can determine $\mathrm{Hom}_R(M, N)$ by using the left exactness of Hom:

$$0 \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}(F, N) \xrightarrow{\phi^*} \mathrm{Hom}(G, N).$$

Therefore $\mathrm{Hom}_R(M, N)$ is the kernel of the map $F^* \otimes_R N \longrightarrow G^* \otimes_R N$, which may be computed using syzygies (as in 3.4).

Let's perform these steps by hand via *Macaulay2* to find the module $\mathrm{Hom}_S(J, S/J)$ where J is the ideal of three lines in $\mathbb{P}^3$.

```
i56 : use S

o56 = S

o56 : PolynomialRing

i57 : J = intersect(ideal(a,b),ideal(a,c),ideal(b,d))

o57 = ideal (a*d, b*c, a*b)

o57 : Ideal of S

i58 : phi = presentation module J

o58 = {2} | b  0  |
      {2} | 0  a  |
      {2} | -d -c |

               3       2
o58 : Matrix S  <--- S

i59 : phi' = (transpose phi) ** S^1/J

o59 = {-3} | b 0 -d |
      {-3} | 0 a -c |

o59 : Matrix

i60 : H = kernel phi'

o60 = subquotient ({-2} | 0 c d 0 0 a |, {-2} | ad bc ab 0  0  0  0  0 ···
                   {-2} | d 0 0 b c 0 |  {-2} | 0  0  0  ad bc ab 0  0 ···
                   {-2} | 0 0 b 0 a 0 |  {-2} | 0  0  0  0  0  0  ad b ···

                                    3
o60 : S-module, subquotient of S
```

Use `trim` to provide a more efficient representation of the module.

```
i61 : trim H

o61 = subquotient ({-2} | a 0 0 d c 0 |, {-2} | 0  0  ad 0  0  bc 0  0 ···
                   {-2} | 0 c b 0 0 d |  {-2} | 0  ad 0  0  bc 0  0  a ···
                   {-2} | 0 a 0 b 0 0 |  {-2} | ad 0  0  bc 0  0  ab 0 ···

                                 3
o61 : S-module, subquotient of S
```

The same steps may be performed automatically:

```
i62 : N = Hom(J,S^1/J)

o62 = subquotient ({-2} | a 0 0 d c 0 |, {-2} | 0  0  ad 0  0  bc 0  0 ···
                   {-2} | 0 c b 0 0 d |  {-2} | 0  ad 0  0  bc 0  0  a ···
                   {-2} | 0 a 0 b 0 0 |  {-2} | ad 0  0  bc 0  0  ab 0 ···

                                 3
o62 : S-module, subquotient of S
```

Notice that the module N is a subquotient module, generated by the columns of the first matrix. You can see the homomorphisms from these columns. For example, the first column corresponds to the homomorphism sending the first generator of I, ad to a, the other generators bc and ab to 0.

This module N is an S-module, but we know that it is also an $R = S/I$ module. This situation is common. In *Macaulay2*, the corresponding R-module is

```
i63 : NR = N ** R

o63 = cokernel {-1} | 0  d 0  0 0  0 0 b 0 -c 0 0 |
               {-1} | 0  0 -d 0 0  0 b 0 0 0  0 0 |
               {-1} | 0  0 0  c -d 0 0 0 0 0  0 a |
               {-1} | c  0 0  0 0  0 0 0 0 0  a 0 |
               {-1} | -d 0 0  0 0  b 0 0 0 a  0 0 |
               {-1} | 0  0 c  0 b  0 0 0 a 0  0 0 |

                            6
o63 : R-module, quotient of R
```

Given an R-module M, it is also an S-module, and we would like to produce a presentation as a quotient of free S-modules. This may be done using the right exactness of tensor products: If

$$R^b \xrightarrow{\phi} R^a \longrightarrow M \longrightarrow 0$$

is a presentation for the R-module M, lift the entries of the matrix ϕ to S in any way, and let

$$S^b \xrightarrow{\phi} S^a \longrightarrow N \longrightarrow 0$$

be a presentation of the cokernel of this lift of ϕ. Right-exactness of tensor products now says that $M \cong N \otimes_S S/I$.

In our example, we may write `NR` as an S-module, by lifting the presentation matrix to S, and tensoring its cokernel with the module S/I.

```
i64 : NS = (coker lift(presentation NR,S) ** S^1/I)
```

```
o64 = cokernel {-1} | bc-ad c3-bd2 ac2-b2d b3-a2c 0     0      0       ···
               {-1} | 0     0      0       0      bc-ad c3-bd2 ac2-b2d ···
               {-1} | 0     0      0       0      0     0      0       ···
               {-1} | 0     0      0       0      0     0      0       ···
               {-1} | 0     0      0       0      0     0      0       ···
               {-1} | 0     0      0       0      0     0      0       ···

                            6
o64 : S-module, quotient of S
```

This provides a very inefficient presentation. As above, use `trim` to simplify the presentation.

```
i65 : trim NS

o65 = cokernel {-1} | 0  d 0  0 0  0 0 b 0 -c 0 0 0   0   0    0   0   ···
               {-1} | 0  0 -d 0 0  0 b 0 0 0  0 0 c2d 0   0    0   0   ···
               {-1} | 0  0 0  c -d 0 0 0 0 0  0 a -d3 bd2 0    0   b2d ···
               {-1} | c  0 0  0 0  0 0 0 0 0  a 0 0   0   -bd2 b2d 0   ···
               {-1} | -d 0 0  0 0  b 0 0 0 a  0 0 0   0   c2d  0   0   ···
               {-1} | 0  0 c  0 b  0 0 0 a 0  0 0 0   0   0    0   0   ···

                            6
o65 : S-module, quotient of S
```

3.4.2 Ext **modules**

Ext is computed in a similar manner to Hom. To obtain $Ext^i_R(M,N)$, we find a free resolution of M, to at least length $i+1$, then apply $\mathrm{Hom}(-,N)$, and take cohomology.

For example, let's compute $\mathrm{Ext}^2(S/I,S)$ "by hand", using *Macaulay2*, where $I \subset S$ is the rational quartic considered earlier. This is (up to a degree twist) the canonical module of R.

```
i66 : C = res (S^1/I);

i67 : E2 = (ker transpose C.dd_3) / (image transpose C.dd_2)

o67 = subquotient ({-4} | a  b c 0 0  0 |, {-4} | -b2 c a  0 |)
                   {-4} | 0  0 d a b  0 |  {-4} | -ac d b  0 |
                   {-4} | 0  d 0 0 -c a |  {-4} | -bd 0 -c a |
                   {-4} | -d 0 0 c 0  b |  {-4} | -c2 0 -d b |

                               4
o67 : S-module, subquotient of S

i68 : prune E2

o68 = cokernel {-3} | c -d 0 a -b |
               {-3} | d 0  c b 0  |
               {-3} | 0 c  b 0 a  |

                            3
o68 : S-module, quotient of S
```

Or, let *Macaulay2* do it for you.

```
i69 : Ext^2(S^1/I,S)

o69 = cokernel {-3} | c -d 0 a -b |
               {-3} | d 0  c b 0  |
               {-3} | 0 c  b 0 a  |
```

```
                                 3
o69 : S-module, quotient of S
```

3.4.3 Computing $\mathrm{Tor}_i^R(M,N)$

Tor is handled in the same manner as Ext. To compute $\mathrm{Tor}_i^R(M,N)$, take a free resolution of M, at least to length $i+1$, and then apply $-\otimes_R N$, and take homology. A key property of Tor is that $Tor_i^R(M,N) = Tor_i^R(N,M)$, so that we could compute this Tor by starting with a free resolution of N instead.

```
i70 : k = coker vars S

o70 = cokernel | a b c d |

                                 1
o70 : S-module, quotient of S

i71 : prune Tor_2(S^1/I,k)

o71 = cokernel {4} | 0 0 0 d 0 0 0 c 0 0 0 b 0 0 0 a |
               {4} | 0 0 d 0 0 0 c 0 0 0 b 0 0 0 a 0 |
               {4} | 0 d 0 0 0 c 0 0 0 b 0 0 0 a 0 0 |
               {4} | d 0 0 0 c 0 0 0 b 0 0 0 a 0 0 0 |

                                 4
o71 : S-module, quotient of S
```

This is the graded vector space k^4, whose 4 generators are all in degree 4.

4 Coherent sheaves and graded modules

In this section, we recall the relationship between coherent sheaves on projective space and graded S-modules. As an important exercise, we consider the computation of the space of global sections of a coherent sheaf.

We will restrict ourselves to graded S-modules M which are either finitely generated, or, eventually finitely generated, that is, some truncation $M_{\geq d}$ is a finitely generated S-module. These will correspond to coherent sheaves on $\mathbb{P}^n$. We could loosen these requirements, and deal with quasicoherent sheaves. For simplicity we will only consider the coherent case.

Definition 4.1 (Construction of $\widetilde{M}$)**.** *Given a graded S-module M which is eventually finitely generated, we will associate to it a coherent sheaf $\widetilde{M}$ of $\mathcal{O}_{\mathbb{P}^n}$-modules. To do so, we will assume the following facts:*

- The standard open sets U_i are an open affine cover of $\mathbb{P}^n$, with affine intersections $U_i \cap U_j$.
- On an affine scheme U, a coherent sheaf $\mathcal{F}$ is completely specified by its $\mathcal{O}_U(U)$-module of global sections $\mathcal{F}(U)$.
- To specify a sheaf on an arbitrary scheme it suffices to specify sheaves $\mathcal{F}_i$ on each open set in an open cover $\{U_i\}$, and compatible isomorphisms

$$\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \longrightarrow \mathcal{F}_j|_{U_i \cap U_j}$$

(see Ex. II.1.23 in Hartshorne [7]).

With this in mind, we define $\widetilde{M}$ as the sheaf on $\mathbb{P}^n$ obtained by glueing the sheaves

$$\widetilde{M}(U_i) := (M \otimes_S S[x_i^{-1}])_0,$$

(the 0 subscript means the subset of all elements of degree 0), via the maps

$$(\widetilde{M}|_{U_i})_{U_i \cap U_j} = (M \otimes S[x_i^{-1}] \otimes S[x_j^{-1}])_0 \longrightarrow (M \otimes S[x_j^{-1}] \otimes S[x_i^{-1}])_0 = (\widetilde{M}|_{U_j})_{U_i \cap U_j}$$

Our hypotheses on M imply that $\widetilde{M}(U_i)$ is a finitely generated $\mathcal{O}(U_i)$-module.

Proposition 4.2. *Some basic properties of this construction include*

(a) $\widetilde{S} = \mathcal{O}_{\mathbb{P}^n}$.

(b) The operation $\widetilde{\ }$ is an exact functor from the category of eventually finitely generated graded S-modules to the category of coherent $\mathcal{O}_{\mathbb{P}^n}$-modules.

(c) If M is an eventually f.g. graded S-module such that $M_d = 0$ for all $d \gg 0$, then $\widetilde{M} = 0$.

(d) Every coherent $\mathcal{O}_{\mathbb{P}^n}$-module is of the form $\widetilde{M}$, for some finitely generated graded S-module M.

We will use part (d) as our definition of coherent sheaf on $\mathbb{P}^n$.

If M and N are eventually finitely generated graded S-modules, we can define an equivalence relation $M \equiv N$ iff there exists a $d \in \mathbb{Z}$ such that $M_{\geq d} \cong N_{\geq d}$. Let $\mathcal{C}$ be the category whose objects are the equivalence classes of this relation.

Definition 4.3. *Given a coherent $\mathcal{O}_{\mathbb{P}^n}$-module, $\mathcal{F}$, define*

$$H^0_*(\mathbb{P}^n, \mathcal{F}) := \bigoplus_{d \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(d)).$$

This is an eventually finitely generated graded S-module. If $\mathcal{F}$ is the extension by zero of a sheaf on $X = V(I) \subset \mathbb{P}^n$, then this module is also a graded $R = S/I$-module.

This operation $H^0_*(-) : CoherentSheaves(\mathbb{P}^n) \longrightarrow \mathcal{C}$ is *not* an exact functor (otherwise a significant portion of these notes would not be necessary!).

Proposition 4.4. *Some basic properties of this construction include*

(a) $H^0_*(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = S(d)$,

(b) For every coherent $\mathcal{O}$-module $\mathcal{F}$,

$$\widetilde{H^0_*(\mathbb{P}^n, \mathcal{F})} \cong \mathcal{F},$$

(c) If M is a graded S-module, then the natural map

$$M \longrightarrow H^0_*(\widetilde{M})$$

is an isomorphism in all degrees $d \gg 0$.

(d) The operations $\widetilde{\ }$ and $H^0_*(-)$ provide an equivalence of categories between $\mathcal{C}$ and the category of coherent sheaves on $\mathbb{P}^n$.

Exercise 4.5. Suppose that $X = \mathbb{P}^m \times \mathbb{P}^n$. X comes equipped with locally free sheaves $\mathcal{O}_X(a,b)$, for $a, b \in \mathbb{Z}$. Let $S = k[x_0, \ldots, x_m, y_0, \ldots, y_n]$. S has a bigrading: $\deg x_i = (1,0)$ and $\deg y_j = (0,1)$.

(a) Given a bigraded S-module M, construct a coherent sheaf $\widetilde{M}$ on X.

(b) Given a coherent sheaf $\mathcal{F}$ on X, define a bigraded S-module as

$$H^0_*(X, \mathcal{F}) = \bigoplus_{a,b \in \mathbb{Z}} H^0(X, \mathcal{F} \otimes \mathcal{O}_X(a,b)).$$

Determine the analogues of the previous two propositions, and the equivalence of categories, in this new situation.

(c) Now generalize this to a product of any number of projective spaces.

4.1 Global sections of a coherent sheaf

Let M be a finitely generated graded S-module. For $d \geq 0$, let $J_d = (x_0^d, \ldots, x_n^d) \subset S$. The sheaf axiom ensures that specifying a global section of a sheaf is equivalent to giving sections on each open set of some open cover, which agree on intersections. Therefore, a global section m of $\widetilde{M}$ is determined by an $n+1$-tuple $(\frac{m_0}{x_0^d}, \ldots, \frac{m_n}{x_n^d})$ of elements of degree zero, such that $\frac{m_i}{x_i^d} = \frac{m_j}{x_j^d}$ in $(M \otimes S[x_i^{-1} x_j^{-1}])_0$.

Exercise 4.6. *Show that every such global section determines an element of $\mathrm{Hom}_S(J_e, M)_0$, for some integer $e \geq d$. Conversely, show that an element of this module defines a global section of $\widetilde{M}$.*

Exercise 4.7. *Show that there are natural maps $\mathrm{Hom}_S(J_d, M) \longrightarrow \mathrm{Hom}_S(J_e, M)$, for $d \leq e$. Conclude that*

$$H^0(\mathbb{P}^n, \widetilde{M}) = \lim_{\ell \to \infty} \mathrm{Hom}_S(J_\ell, M)_0,$$

and that

$$H^0_*(\mathbb{P}^n, \widetilde{M}) = \lim_{\ell \to \infty} \mathrm{Hom}_S(J_\ell, M).$$

Exercise 4.8. *Show that, given M, there is a bound a, such that for all $\ell \geq a$,*

$$H^0(\mathbb{P}^n, \widetilde{M}) = \mathrm{Hom}_S(J_\ell, M)_0.$$

Hint: First prove this for graded free modules, and then consider the free resolution of M.

For example, let's find $H^0(\mathbb{P}^3, \mathcal{O}_X(1))$, where $X \subset \mathbb{P}^3$ is the rational quartic curve.

```
i72 : M = (S^1/I) ** S^{1}

o72 = cokernel {-1} | bc-ad c3-bd2 ac2-b2d b3-a2c |

                               1
o72 : S-module, quotient of S

i73 : hilbertFunction(0,M)

o73 = 4

i74 : M1 = prune Hom(image vars S, M)

o74 = cokernel {0}  | -c -d 0     -b -a |
               {-1} | bd c2 bc-ad ac b2 |

                               2
o74 : S-module, quotient of S

i75 : hilbertFunction(0,M1)

o75 = 5

i76 : use S

o76 = S

o76 : PolynomialRing

i77 : M2 = prune Hom(ideal(a^2,b^2,c^2,d^2), M)

o77 = cokernel {0}  | -c -d 0     -b -a |
               {-1} | bd c2 bc-ad ac b2 |

                               2
o77 : S-module, quotient of S
```

This is the same module as M_1.

```
i78 : M1 == M2

o78 = true
```

It turns out that in this case, the dimension of $H^0(\mathcal{O}_X(1))$ has already been reached. Try proving this. M and M_1 define the same sheaf $\mathcal{O}_X(1)$, and $H^0(\mathcal{O}_X(1))$ has dimension five.

4.2 Improving the presentation of a sheaf

If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$, there are many (i.e. an infinite number of) graded S-modules M such that $\mathcal{F} \cong \widetilde{M}$. Some of these modules may have small complexity: short resolutions, nice generators, etc, while others may be given with a large number of generators.

Here is the problem we wish to address: given a graded S-module M, find another (finitely generated) graded S-module N which is "nicer" in some way, and for which $\widetilde{M} \cong \widetilde{N}$.

The canonical choice would be the module $H^0_*(\widetilde{M})$. Unfortunately, this is sometimes infinitely generated. The following proposition tells us that in certain common circumstances, this module will be finitely generated. The proposition is a simple corollary of local duality, which we will prove later (see Proposition 6.5).

Proposition 4.9. *Let M be a finitely generated $S = k[x_0, \dots, x_n]$-module.*

(a) $M = H^0_*(\widetilde{M})$ if and only if $pdim_S(M) \leq n - 1$.

(b) $H^0_*(\widetilde{M})$ is a finitely generated S-module if and only if $Ext^n_S(M, S)$ has finite dimension as a k-vector space.

(c) If $X = V(I)$ is a smooth variety of positive dimension (more generally, a locally Cohen-Macaulay scheme of positive dimension), then $H^0_*(\mathcal{O}_X)$ is a finitely generated S-module.

We can use exercise 4.8 to find this module.

For example, consider $X = V(I) \subset \mathbb{P}^n$ defined by an ideal I. The *saturation* I^{sat} of I is defined to be

$$I^{sat} := (I : (x_0, \dots x_n)^\infty) = \{f \in S \mid x_i^d f \in S, \text{for all } 0 \leq i \leq n, \text{ and } d \gg 0\}.$$

Exercise 4.10. *Show that $\widetilde{I} \cong \widetilde{I^{sat}}$, and $\widetilde{S/I} \cong \widetilde{S/I^{sat}}$.*

Similarly, if $I \subset F$ is an S-submodule of a graded free S-module F, we may define $I^{sat} \subset F$ in a similar manner ($I^{sat} := (I : (x_0, \dots, x_n)^\infty) \subset F$), and you should check that $\widetilde{I} \cong \widetilde{I^{sat}}$, and $\widetilde{F/I} \cong \widetilde{F/I^{sat}}$.

4.3 Operations on coherent sheaves

There is a dictionary between operations on sheaves on $\mathbb{P}^n$, and graded S-modules. We describe a few of these dictionary items here.

Proposition 4.11. *Let M and N be graded S-modules.*

(a) $\widetilde{M} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \widetilde{N} \cong \widetilde{M \otimes_S N}$.

(b) $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\mathrm{Hom}_S(M, N)}$.

We will see many different uses of these constructions later, including the group operations for $Pic(X)$, and the normal sheaf.

4.4 Fitting ideals and locally free sheaves

If $\mathcal{F}$ is a locally free sheaf on X of rank r, then $\mathcal{F}$ is the sheaf of sections of a well-defined vector bundle. It is common practice in algebraic geometry to abuse language and call this sheaf $\mathcal{F}$ a vector bundle. Similarly, if $\mathcal{F}$ is locally free of rank 1, we will call $\mathcal{F}$ a line bundle on X. Vector bundles are among the most important sheaves on a variety X.

Using Fitting ideals one can detect whether a sheaf $\widetilde{M}$ on $X \subset \mathbb{P}^n$ is a vector bundle.

Definition 4.12 (Fitting ideal)**.** *Suppose that the finitely generated R-module M has a presentation*

$$R^a \xrightarrow{\phi} R^b \longrightarrow M \longrightarrow 0.$$

The i-th Fitting ideal*, $Fitt_i(M)$, is the ideal of R generated by the $(b-i) \times (b-i)$ minors of the matrix ϕ.*

The main properties of Fitting ideals are (see Eisenbud [2] for a more detailed treatment):

- The Fitting ideals are independent of the given presentation of M.
- $V(Fitt_i(M)) \subset X$ is the locus of points $p \in X$ such that M is not locally generated by i elements.
- If $Fitt_{r-1}(M) = 0$, then the locus of points $p \in X$ where M is not locally free of rank r is $V(Fitt_r(M)) \subset X$.

A projective variety or scheme X is a local complete intersection if the ideal I of X is locally generated by codim(X) elements. This is the same as saying that the conormal sheaf $\widetilde{I/I^2}$ is a vector bundle of rank codim(X). Every smooth variety is a local complete intersection.

As an example, let's check that the rational quartic is a local complete intersection. Since the rational quartic has codimension two, this means that the ideal of the quartic should be locally generated by two elements.

```
i79 : phi = presentation module I

o79 = {2} | ac c2 b2 bd |
      {3} | 0  -b 0  -a |
      {3} | -b d  -a c  |
      {3} | -d 0  -c 0  |

               4       4
o79 : Matrix S  <--- S
```

I is generated by ≤ 4 equations everywhere. Off of X, I is the unit ideal. The locus of points of $\mathbb{P}^3$ where I is not generated by ≤ 1 equations is X:

```
i80 : trim minors(3,phi)

                          2     2            2                    2    3  ···
o80 = ideal (b*c*d - a*d , b*c  - a*c*d, b c - a*b*d, a*b*c - a d, c d ···

o80 : Ideal of S

i81 : radical trim minors(3,phi)

                           3      2     2    2    3    2
o81 = ideal (b*c - a*d, c  - b*d , a*c  - b d, b  - a c)

o81 : Ideal of S
```

The locus of points of X where I is not generated by ≤ 2 equations is

```
i82 : trim minors(2,phi)

               2                  2             2        2
o82 = ideal (d , c*d, b*d, a*d, c , b*c, a*c, b , a*b, a )

o82 : Ideal of S
```

This defines the empty set in $X \subset \mathbb{P}^3$, so I is a local complete intersection of codimension 2. One may use the `fittingIdeal` routine in *Macaulay2*.

```
i83 : fittingIdeal(2,module I)

               2                  2             2        2
o83 = ideal (d , c*d, b*d, a*d, c , b*c, a*c, b , a*b, a )
```

```
o83 : Ideal of S
```

We test whether the conormal sheaf $\widetilde{I/I^2}$ on X is locally free of rank 2. We first create the R-module $I/I^2 \cong I \otimes_S S/I$:

```
i84 : N' = (module I) ** R

o84 = cokernel {2} | ac c2 b2 bd |
               {3} | 0  -b 0  -a |
               {3} | -b d  -a c  |
               {3} | -d 0  -c 0  |

                                4
o84 : R-module, quotient of R

i85 : fittingIdeal(1,N')

o85 = 0

o85 : Ideal of R

i86 : fittingIdeal(2,N')

              2                   2        2        2
o86 = ideal (d , c*d, b*d, a*d, c , a*c, b , a*b, a )

o86 : Ideal of R
```

Since the zero set of this ideal is the empty set in $X \subset \mathbb{P}^3$, $\widetilde{I/I^2}$ is locally free of rank 2 on X.

On the other hand, the conormal sheaf of the union of two planes in $\mathbb{P}^4$ meeting at a point is not locally free:

```
i87 : S1 = kk[a..e];

i88 : I1 = intersect(ideal(a,b),ideal(c,d))

o88 = ideal (b*d, a*d, b*c, a*c)

o88 : Ideal of S1

i89 : R1 = S1/I1;

i90 : N' = (module I1) ** R1

o90 = cokernel {2} | 0  c  0  a  |
               {2} | c  0  0  -b |
               {2} | 0  -d a  0  |
               {2} | -d 0  -b 0  |

                                  4
o90 : R1-module, quotient of R1

i91 : fittingIdeal(1,N')

o91 = 0

o91 : Ideal of R1

i92 : fittingIdeal(2,N')

              2        2   2        2
o92 = ideal (d , c*d, c , b , a*b, a )

o92 : Ideal of R1
```

Therefore, $\widetilde{I_1/I_1^2}$ is locally free of rank 2 everywhere except at the point $(0,0,0,0,1)$ of intersection of the two planes.

4.5 Hilbert polynomials and the Euler characteristic

Definition 4.13 (Euler characteristic). *Given a coherent sheaf $\mathcal{F}$ on $X \subset \mathbb{P}^n$, the* Euler characteristic $\chi(\mathcal{F})$ *of $\mathcal{F}$ is*

$$\chi(\mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i dim H^i(\mathcal{F}).$$

Exercise 4.14. *Show that $d \mapsto \chi(\widetilde{M}(d))$ is a polynomial function, and that this polynomial is the Hilbert polynomial $P_M(d)$ of M. Consequently show that $\chi(\widetilde{M}) = P_M(0)$.*

As a simple example, the Hilbert polynomial of the coordinate ring of the rational quartic is:

```
i93 : P = hilbertPolynomial(S^1/I)

o93 = - 3*P  + 4*P
           0      1

o93 : ProjectiveHilbertPolynomial
```

The default in *Macaulay2* is to write the Hilbert polynomial as a sum of Hilbert polynomials of projective spaces. The intuition here is that, numerically, the rational quartic X is like the union of 4 lines, with three intersection points. The actual polynomial is:

```
i94 : P1 = hilbertPolynomial(S^1/I, Projective=>false)

o94 = 4i + 1

o94 : QQ [i]
```

The euler characteristic is the value of P at 0:

```
i95 : (P(0), euler(S^1/I))

o95 = (1, 1)

o95 : Sequence
```

5 Sheaves in nature

We consider many useful examples of coherent sheaves, including locally free sheaves, the cotangent bundle, divisors and line bundles, the canonical divisor, and several others.

5.1 Divisors and line bundles

For now, let's suppose that $X = V(I) \subset \mathbb{P}^n$, is a smooth projective variety (or, perhaps only a normal projective variety). Recall that a divisor D on X

determines a locally free sheaf $\mathcal{O}_X(D)$ of rank 1 on X (i.e. a line bundle). Two divisors are linearly equivalent if and only if their corresponding line bundles are isomorphic.

The set of isomorphism classes of line bundles on X form a group, the *Picard* group $Pic(X)$ of X, with group operation given by

$$\mathcal{O}_X(D) \bullet \mathcal{O}_X(E) := \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$$

and inverse given by

$$\mathcal{O}_X(D)^{-1} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X).$$

We can perform these operations effectively using proposition 4.11.

Proposition 5.1. *Let $R = S/I$ be the homogeneous coordinate ring of X. If $\mathcal{O}_X(D) = \widetilde{M}$, and $\mathcal{O}_X(E) = \widetilde{N}$, then*

(a) $\mathcal{O}_X(D+E) = \widetilde{M \otimes_R N}$,

(b) $\mathcal{O}_X(-D) = \widetilde{\mathrm{Hom}_R(M,R)}$,

(c) $\mathcal{O}_X(D-E) = \widetilde{\mathrm{Hom}_R(N,M)}$.

It is easy to produce such line bundles in the first place, because if $J \subset R$ is the ideal of an effective divisor D, then $\mathcal{O}_X(-D) = \widetilde{J}$.

It should be noted that the module M only determines the divisor D up to linear equivalence. However, it is possible to recover a divisor D such that $\widetilde{M}$ is isomorphic to $\mathcal{O}_X(D)$:

Proposition 5.2. *If $\widetilde{M}$ is a locally free sheaf of rank one on X, then there exists an ideal $J \subset R$ and an integer $d \in \mathbb{Z}$ such that*

$$\widetilde{M} \cong \widetilde{J(d)}$$

In this case, if $H = V(x_0) \subset X$ is the hyperplane section, then $\widetilde{M} = \widetilde{J(d)}$ is isomorphic to $\mathcal{O}_X(dH - V(J))$.

Proof. Consider the R-module $M^* = \mathrm{Hom}_R(M,R)$. Since M is locally free, M^* cannot be the zero module. Choose a non-zero $f \in M^*$ of degree d. Then $f : M \to R(d)$ is a degree 0 homomorphism, and its kernel must be supported at the ideal $(x_0, \ldots, x_n)$. Therefore if $J(d) \subset R(d)$ is the image of f, then $\widetilde{M} \cong \widetilde{J(d)}$. □

Example 5.3 (Canonical sheaf as an ideal)**.** The most important line bundle on X is the canonical sheaf ω_X. We describe how to find ω_X in section 5.6 later.

For now let us simply assume that for the rational quartic curve, the canonical sheaf corresponds to the R-module $\mathrm{Ext}^2_S(S/I, S(-4))$.

```
i96 : KX = Ext^2(S^1/I,S^{-4}) ** R

o96 = cokernel {1} | c -d 0 a -b |
               {1} | d 0  c b 0  |
               {1} | 0 c  b 0 a  |

                              3
o96 : R-module, quotient of R

i97 : KXdual = Hom(KX,R)

o97 = image {-1} | -cd c2  -ad -ac bd  -b2 -ab |
            {-1} | c2  -bd ac  b2  -ad ab  a2  |
            {-1} | -d2 cd  -bd -ad c2  -ac -b2 |

                               3
o97 : R-module, submodule of R
```

Each (generalized) column corresponds to a homomorphism. *Macaulay2* provides a mechanism to get the homomorphisms corresponding to (combinations of) generators. For example, the first generator of H:

```
i98 : KXdual_{0}

o98 = {1} | 1 |
      {1} | 0 |
      {1} | 0 |
      {1} | 0 |
      {1} | 0 |
      {1} | 0 |
      {1} | 0 |

o98 : Matrix
```

The corresponding homomorphism from KX to R:

```
i99 : f = homomorphism KXdual_{0}

o99 = | -cd c2 -d2 |

o99 : Matrix

i100 : ker f

o100 = subquotient ({1} | c -d 0 a -b |, {1} | c -d 0 a -b |)
                    {1} | d 0  c b 0  |  {1} | d 0  c b 0  |
                    {1} | 0 c  b 0 a  |  {1} | 0 c  b 0 a  |

                                  3
o100 : R-module, subquotient of R

i101 : J = ideal image f

                      2    2
o101 = ideal (-c*d, c , -d )

o101 : Ideal of R

i102 : degree f

o102 = {1}

o102 : List
```

The ideal J is supported at the point $p = (1, 0, 0, 0)$, and has multiplicity 6.

```
i103 : degree(R^1/J)

o103 = 6
```

Therefore, $K_X \cong J(1)$, and $\omega_X \cong \mathcal{O}_X(H - 6p)$. Since $\mathcal{O}_X(H) \cong \mathcal{O}_X(4p)$, this says that $\omega_X \cong \mathcal{O}_X(-2p)$, as expected, since the canonical sheaf on $\mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(-2)$.

5.1.1 The degree of a line bundle on a curve

Suppose that X is a smooth projective curve, and $\mathcal{L} = \widetilde{M}$ is a line bundle. We know how to compute the euler characteristic of $\mathcal{L}$: if the Hilbert polynomial of M is $P(d)$, then $\chi(\mathcal{L}) = P(0)$. As a result, the Riemann-Roch theorem for curves gives us the degree of a line bundle:

Proposition 5.4.

$$\deg \mathcal{L} = \chi(\mathcal{L}) - \chi(\mathcal{O}_X)$$

Since the rational quartic curve is rational, the degree of the canonical bundle should be -2. Let's check this.

```
i104 : degKX = euler KX - euler(S^1/I)

o104 = -2
```

5.1.2 Intersection numbers on a surface

Let $X \subset \mathbb{P}^n$ be a smooth projective surface. One way to define intersection numbers for line bundles on X is the following:

Definition 5.5. *If $\mathcal{O}_X(D)$ and $\mathcal{O}_X(E)$ are both line bundles on X, define*

$$D \cdot E := \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D)) - \chi(\mathcal{O}_X(-E)) + \chi(\mathcal{O}_X(-D-E)).$$

It is worth mentioning that if D and E are different codimension one irreducible subvarieties of X, then $D \cdot E$ is the number of intersection points of D and E, counted with multiplicity.

For the next example, let's define a *Macaulay2* function for this intersection number, given R-modules corresponding to the two sheaves.

```
i105 : intersectionNumber = (M,N) ->
           euler ring M - euler M - euler N + euler(M ** N)

o105 = intersectionNumber

o105 : Function
```

Example 5.6 (Intersection theory on a cubic surface)**.** Let $X = V(F) \subset \mathbb{P}^3$ be a smooth cubic surface. There are exactly 27 lines on this surface.

```
i106 : use S;
```

The following numbers are the elementary symmetric functions of $1, 2, 3, 4, 5$.

```
i107 : (s1,s2,s3,s4,s5) = (15,85,225,274,120);
```

Exercise 7.9 in Mile's Reid's book [11] shows that the following cubic surface has all of its 27 lines defined over the rationals. (Find them!)

```
i108 : F = a^2*c-b^2*d + a*(s5*d^2+s3*c*d+s1*c^2) - b*(s4*d^2+s2*c*d+c^2)

        2          2       2    2
o108 = a c + 15a*c  - b*c  - b d + 225a*c*d -
       -----------------------------------------
                       2         2
       85b*c*d + 120a*d  - 274b*d

o108 : S

i109 : RF = S/F;
```

Consider the lines $L_1 = V(a, b)$, $L_2 = V(c, d)$. Their modules are

```
i110 : L1 = Hom(ideal(a,b),RF)

o110 = image {-1} | a c2+bd+85cd+274d2    |
             {-1} | b ac+15c2+225cd+120d2 |

                                         2
o110 : RF-module, submodule of RF

i111 : L2 = Hom(ideal(c,d),RF)

o111 = image {-1} | c b2-120ad+274bd        |
             {-1} | d a2+15ac-bc+225ad-85bd |

                                         2
o111 : RF-module, submodule of RF
```

L_1 and L_2 do not meet on X.

```
i112 : intersectionNumber(L1,L2)

o112 = 0
```

Since there are only finitely many lines on X, each line must have negative self-intersection.

```
i113 : intersectionNumber(L1,L1)

o113 = -1

i114 : intersectionNumber(L2,L2)

o114 = -1
```

5.2 Sheaves of differentials

If $\mathbb{A}^n$ is affine n-space over k, then the module (equivalently, the sheaf) of differentials $\Omega^1_{\mathbb{A}^n}$ is the free $A = k[x_1, \ldots, x_n]$-module with generators $dx_1, \ldots, dx_n$. If $X = V(J) \subset \mathbb{A}^n$ is an affine variety, then the module of differentials Ω^1_X is the quotient of the free A/J-module generated by $dx_1, \ldots, dx_n$, by elements $\{df \mid f \in I\}$:

$$I/I^2 \longrightarrow \Omega^1_{\mathbb{A}^n} \otimes A/J \longrightarrow \Omega^1_X \longrightarrow 0$$

For $\mathbb{P}^n$, the sheaf of differentials $\Omega^1_{\mathbb{P}^n}$ should be defined so that on each standard open set U_i, the restriction is $\Omega^1_{U_i}$ defined above, and that on $U_i \cap U_j$ we can glue the two resulting definitions. It is a worthwile exercise to check that there is a very efficient description of the resulting sheaf as the kernel of the first map $d_1 = (x_0, \ldots, x_n)$ in the Koszul complex:

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \mathcal{O}^{n+1}_{\mathbb{P}^n}(-1) \xrightarrow{d_1} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

If $X \subset \mathbb{P}^n$ is a subvariety or subscheme, another good exercise is to check that

$$I_X/I_X^2 \longrightarrow \Omega^1_{\mathbb{P}^n} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

More explicitly, we have:

Proposition 5.7 (Cotangent sheaf)**.** *Let $X = V(I) \subset \mathbb{P}^n$. Let $R = S/I$. The cotangent sheaf of X is the sheaf associated to the homology module of*

$$F \otimes R \xrightarrow{dj} R(-1)^{n+1} \xrightarrow{d_1} R,$$

where if $j : F \longrightarrow S$ is the generator matrix of the ideal I, then $dj : F \otimes R \longrightarrow R(-1)^{n+1}$ is the jacobian matrix of j, and the second map $d_1 = (x_0, \ldots, x_n)$.

5.3 Differential p-forms

The sheaf of differential p-forms on X is defined to be the sheaf

$$\Omega^p_X := \wedge^p \Omega^1_X.$$

Let's start with projective space again, as there is a nice presentation for this.

Proposition 5.8. *There is an exact sequence*

$$0 \longrightarrow \Omega^p_{\mathbb{P}^n} \longrightarrow (\wedge^p \mathcal{O}^{n+1})(-p) \longrightarrow \Omega^{p-1} \longrightarrow 0.$$

Therefore, $\Omega^p \cong \ker d_p$, where d_p is the map of sheaves corresponding to the pth map in the Koszul complex.

Exercise 5.9. *Find the sheaf cohomology $H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(d))$, for all p, q, and d.*

Given any sheaf $\widetilde{M}$, we may find a presentation of $\wedge^p \widetilde{M}$ from the following proposition. Usually the module that we will obtain is fairly nasty. It is an interesting open problem to find (in a reasonable way) more efficient presentations for the sheaf.

Proposition 5.10 (p-th exterior power of a sheaf)**.** *Let $G \longrightarrow F \longrightarrow M \longrightarrow 0$ be a presentation for the R-module M, where F and G are (graded) free R-modules. The p-th exterior power $\wedge^p M$ of M has presentation*

$$G \otimes \wedge^{p-1} F \longrightarrow \wedge^p F \longrightarrow \wedge^p M \longrightarrow 0.$$

As a simple example, let's consider the sheaf of 1-forms and 2-forms on $\mathbb{P}^2$.

```
i115 : S3 = kk[a,b,c];

i116 : C = res coker vars S3

          1       3       3       1
o116 = S3  <-- S3  <-- S3  <-- S3  <-- 0

       0       1       2       3       4

o116 : ChainComplex
```

Ω^1 is the kernel of d_1, therefore the image of d_2, and therefore the cokernel of d_3:

```
i117 : Omega1 = cokernel C.dd_3

o117 = cokernel {2} | c  |
                {2} | -b |
                {2} | a  |

                                 3
o117 : S3-module, quotient of S3
```

$\Omega^2 \cong \mathcal{O}_{\mathbb{P}^2}(-3)$. The following presentation doesn't make this immediately obvious.

```
i118 : exteriorPower(2,Omega1)

o118 = cokernel {4} | b  c  0  |
                {4} | -a 0  c  |
                {4} | 0  -a -b |

                                 3
o118 : S3-module, quotient of S3
```

We can clean up the presentation, finding that $\Omega^2 = \mathcal{O}(-3)$:

```
i119 : HH^0((sheaf oo)(>=0))

          1
o119 = S3

o119 : S3-module, free, degrees {3}
```

We will not need it here, but we should mention that there is a simple analogue for symmetric powers:

Proposition 5.11 (p-th symmetric power of a sheaf)**.** *Let $G \longrightarrow F \longrightarrow M \longrightarrow 0$ be a presentation for the R-module M, where F and G are (graded) free R-modules. The p-th symmetric power S_pM of M has presentation*

$$G \otimes S_{p-1}F \longrightarrow S_pF \longrightarrow S_pM \longrightarrow 0.$$

For a discussion of these presentations, see [2], Proposition A.2.2.

5.4 Example: The Hodge diamond

Given a smooth projective variety $X \subset \mathbb{P}^n$ of dimension d, the *Hodge diamond* is the set of numbers $h^{p,q}(X) := h^q(X, \Omega^p_X)$. There are several relationships which follow from duality: $h^{p,q}(X) = h^{d-p,d-q}(X)$, and $h^{p,q}(X) = h^{q,p}(X)$. The

Hodge decomposition shows that, if $k = \mathbb{C}$, the singular cohomology groups satisfy

$$\dim H^i(X,\mathbb{C}) = \bigoplus_{p+q=i} h^{p,q}(X).$$

Basically, this set of numbers is a good initial picture of a variety, and contains a wealth of information. For more information about the Hodge decomposition, see Chapter 0 in Griffiths-Harris [6].

Example 5.12 (Hodge diamond for a cubic threefold)**.** Let $X = V(x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3) \subset \mathbb{P}^4$ be the Fermat cubic threefold.

```
i120 : P4 = kk[x_0..x_4];

i121 : IX = ideal(x_0^3+x_1^3+x_2^3+x_3^3+x_4^3);

o121 : Ideal of P4

i122 : X = variety IX;

i123 : time matrix table(4,4,(p,q) ->
          rank HH^q(exteriorPower(p,cotangentSheaf X)))
     -- used 53.88 seconds

o123 = | 1 0 0 0 |
       | 0 1 5 0 |
       | 0 5 1 0 |
       | 0 0 0 1 |

                4          4
o123 : Matrix ZZ  <--- ZZ
```

It is clear that we did too much work here, as we didn't use any of the dualities, or the Euler characteristic. In a project/exercise later in these notes, we will apply the techniques for computing sheaf cohomology that we learn to find faster methods to find this diamond.

5.5 The normal and conormal sheaves

Normal vectors at $p \in X \subset \mathbb{P}^n$ are tangent vectors to $\mathbb{P}^n$, modulo vectors tangent to X at p. At every point $p \in \mathbb{P}^n$, the tangent space $T_p(\mathbb{P}^n) = T_p(X) \oplus E$, Thus $E = T_p(\mathbb{P}^n)/T_p(X)$. The normal bundle is obtained by performing this operation at all points simultaneously: If $X \subset \mathbb{P}^n$ is smooth, and if $\mathcal{T}_X = (\Omega^1_X)^*$ is the tangent sheaf, then:

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}^n} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_X \to \mathcal{N}_{X/\mathbb{P}^n} \to 0.$$

If X is not smooth, defining the normal sheaf as the dual of the conormal sheaf $\widetilde{I/I^2}$ turns out to be more useful:

Definition 5.13. *Let $X = V(I) \subset Y = V(J) \subset \mathbb{P}^n$. Let $R = S/I$ be the homogeneous coordinate ring of X. The conormal sheaf of X in Y is the sheaf associated to the graded R-module $I/(J + I^2)$. The normal sheaf of X in Y is*

$$\mathcal{N}_{X/Y} = \mathrm{Hom}_R(\widetilde{I/(J + I^2)}, R).$$

Note that the normal sheaf may also be defined to be the sheaf associated to the S-module $\mathrm{Hom}_S(I/J, S/I)$. Both the normal and conormal sheaves are locally free if X is a local complete intersection in Y.

5.6 The canonical sheaf

If $X \subset \mathbb{P}^n$ is a smooth variety of dimension d, then the canonical bundle ω_X is the line bundle $\wedge^d \Omega^1_X$. A canonical divisor K_X is the divisor of any meromorphic (rational) d-form on X. Thus $\omega_X = \mathcal{O}_X(K_X)$.

One may relax the hypotheses about smoothness of X, and then ω_X may not be a line bundle. For now, we will not worry about these extensions.

The following proposition follows from Serre duality.

Proposition 5.14. *Suppose that $X \subset \mathbb{P}^n$ is a smooth variety of codimension c. The canonical sheaf ω_X is given by*

$$\omega_X \;=\; \mathrm{Ext}^c_S(\widetilde{S/I, S(}-n-1)).$$

Proposition 5.15 (Adjunction)**.** *If $X \subset Y \subset \mathbb{P}^n$ are projective smooth varieties,* $\mathrm{codim}(X, Y) = c$*, and $\mathcal{N}_{X/Y}$ is the normal bundle, then the canonical bundle ω_X is*

$$\omega_X \;=\; \wedge^c \mathcal{N}_{X/Y} \otimes_{\mathcal{O}_Y} \omega_Y.$$

If the codimension of X in Y is one, then $\mathcal{N}_{X/Y} = \mathcal{O}_Y(X) \otimes_{\mathcal{O}_Y} \mathcal{O}_X$, and so

$$\omega_X \;=\; \mathcal{O}_Y(X) \otimes_{\mathcal{O}_Y} \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_X.$$

5.6.1 Example: Smoothness on the Hilbert scheme

The Hilbert scheme $\mathcal{H} = \sum_{p(d)} \mathcal{H}_{p(d)}$ is a scheme which parametrizes the set of all saturated ideals $I \subset S = k[x_0, \ldots, x_n]$ having Hilbert polynomial $P_{S/I}(d) = p(d)$. The $\mathcal{H}_{p(d)}$ are projective algebraic sets (schemes). The study of their local geometry is the object of deformation theory.

Groebner bases correspond to paths on $\mathcal{H}$. Given an ideal I, there is a path (i.e. a flat deformation) $\{I_t\}$, such that for $t \neq 0$, I_t is the same as I, except for rescaling the variables, and for $t = 0$, $I_0 = in(I)$.

Proposition 5.16. *Let $I \subset S$ be a homogeneous ideal, and let $X \subset \mathbb{P}^n$ be the corresponding subscheme of projective space. Let $[I] \in \mathcal{H}$ denote the corresponding point on the Hilbert scheme. Let $\mathcal{N} = \mathcal{N}_{X/\mathbb{P}^n}$ be the normal sheaf of X in $\mathbb{P}^n$. Then*

(a) The Zariski tangent space of $\mathcal{H}$ at $[I]$ has dimension $h^0(N_{X/\mathbb{P}^n})$.

(b) If X is a local complete intersection, then every component of $\mathcal{H}$ through the point $[I]$ has dimension at least $h^0(\mathcal{N}) - h^1(\mathcal{N})$.

(c) If $X = V(I)$ is any projective scheme, then every component of $\mathcal{H}$ through the point $[I]$ has dimension at least $h^0(\mathcal{N}) - \dim \mathrm{Ext}^1_X(\widetilde{I}, \mathcal{O}_X)$, where this last Ext *group is global Ext.*

If X is a local complete intersection, then $\mathrm{Ext}^1_X(\widetilde{I}, \mathcal{O}_X) \cong H^1(\mathcal{N})$, so (b) follows from (c). For a proof of this proposition, see [8].

For an example, consider the Hilbert scheme corresponding to the rational quartic curve in $\mathbb{P}^3$. First

```
i124 : N = Hom((module I) ** R,R)

o124 = image {-2} | c   -16001d b   c  a   0   b  |
             {-3} | 2d2 0       2c2 d2 2bd -cd c2 |
             {-3} | bd  16001c2 ac  0  b2  -ad 0  |
             {-3} | 0   ac      0   b2 0   ab  a2 |

                                 4
o124 : R-module, submodule of R

i125 : HH^1(sheaf N)

o125 = 0

o125 : kk-module

i126 : HH^0(sheaf N)

         16
o126 = kk

o126 : kk-module, free
```

Since we know from earlier that $X = V(I)$ is a local complete intersection (all smooth subvarieties of $\mathbb{P}^n$ are), this implies that $[I]$ is a smooth point on its Hilbert scheme, and the dimension of that irreducible component of the Hilbert scheme is 16. Its Groebner basis gives a path on the Hilbert scheme, to its initial ideal

```
i127 : I0 = ideal leadTerm I

                     3     2   3
o127 = ideal (b*c, c , a*c , b )

o127 : Ideal of S

i128 : R0 = S/I0

o128 = R0

o128 : QuotientRing
```

The normal sheaf for the initial ideal is

```
i129 : N0 = Hom((module I0) ** R0, R0)

o129 = image {-2} | 0 c b 0  0  0  0  0  0  |
             {-3} | 0 0 0 0  0  c2 0  0  0  |
             {-3} | 0 0 0 0  c2 0  0  ac b2 |
             {-3} | b 0 0 c2 0  0  ac 0  0  |

                                   4
o129 : R0-module, submodule of R0

i130 : HH^0(sheaf N0)

         20
o130 = kk

o130 : kk-module, free
```

Since the dimension of the Zariski tangent space at $[I_0]$ is $20 > 16$, the Hilbert scheme is singular at $[I_0]$. Most likely, several components of this Hilbert scheme pass through that point.

```
i131 : HH^1(sheaf N0)

o131 = 0

o131 : kk-module
```

The initial ideal is not a local complete intersection, so even though $H^1(\mathcal{N}) = 0$, it doesn't say anything about smoothness at that point.

```
i132 : X0 = Proj R0

o132 = X0

o132 : ProjectiveVariety

i133 : IIX0 = sheaf((module I0) ** R0)

o133 = cokernel {2} | b2 0  ac c2 |
                {3} | 0  a  0  -b |
                {3} | 0  -c -b 0  |
                {3} | -c 0  0  0  |

                                            1            3
o133 : coherent sheaf on X0, quotient of OO   (-2) ++ OO   (-3)
                                           X0           X0
```

The obstruction to smoothness sits in the global Ext vector space $\mathrm{Ext}^1_X(\widetilde{I/I^2}, \mathcal{O}_X)$. In this case, since X is not a local complete intersection, this is not $H^1(\mathcal{N})$.

```
i134 : Ext^1(IIX0, OO_X0)

         7
o134 = kk

o134 : kk-module, free
```

Remember that even for local complete intersections X, $H^1(\mathcal{N}_X) \neq 0$ does not imply that the Hilbert scheme is singular at the point corresponding to X.

Each Hilbert scheme has a canonically defined point: the lexicographic point. This corresponds to the monomial ideal defined by the lexicographically first monomials with the given Hilbert polynomial. A surprising result is that this point is smooth on the Hilbert scheme. See [10] for the statement and proof.

6 Cohomology of sheaves

Given a coherent sheaf $\mathcal{F} = \widetilde{M}$, we wish to compute the S-modules

$$H^i_*(\mathcal{F}) = \bigoplus_{d \in \mathbb{Z}} H^i(\mathcal{F}(d)).$$

To simplify notation, we soemtimes denote these modules by

$$H^i(M) := H^i_*(\widetilde{M}).$$

Lemma 6.1. *Define the complex* $\mathcal{C}$ *to be:*

$$\mathcal{C} : 0 \longrightarrow \bigoplus_{i=0..n} M[x_i^{-1}] \longrightarrow \bigoplus_{i<j} M[x_i^{-1}x_j^{-1}] \longrightarrow \cdots \longrightarrow M[(x_0 \dots x_n)^{-1}] \longrightarrow 0.$$

Then $H^i_*(\widetilde{M}) = H^i(\mathcal{C})$.

This allows us to compute the cohomology of $\mathcal{O}_{\mathbb{P}^n}(d)$, for all d:
Proposition 6.2.

$$H^i_*(\mathcal{O}_{\mathbb{P}^n}) = \begin{cases} S & i = 0 \\ 0 & 1 \le i \le n-1 \\ \frac{1}{x_0 \dots x_n} k[x_0^{-1}, \dots, x_n^{-1}] & i = n \end{cases}$$

Corollary 6.3. *For any integer* d*,*

$$\begin{aligned} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) &= S_d \\ H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) &= 0 \\ H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) &= S'_{-n-1-d}. \end{aligned}$$

Definition 6.4. *Let* M *be a graded* S*-module. Define the graded* k*-dual of* M *to be the graded* S*-module* $M^\vee := \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_k(M_{-d}, k)$.

The operation $M \mapsto M^\vee$ is an exact contravariant functor. Notice that if M is a finitely generated graded S-module, which is not zero in high degrees, then $M^\vee$ is not finitely generated. In the above proposition, note that $H^n_*(\mathcal{O}_{\mathbb{P}^n}) = S(-n-1)^\vee$.

6.1 Local duality

We now prove perhaps the most generally useful formulas for finding specific cohomology modules. This technique doesn't exactly give us the module structure, but it is very useful for relating homological properties of the module M, such as depth and projective dimension, with properties of the cohomology modules.

Theorem 6.5 (Local Duality)**.** *Let* M *be a graded* S*-module. Then*
(a) The following sequence is exact.

$$0 \longrightarrow \mathrm{Ext}^{n+1}_S(M, S(-n-1))^\vee \longrightarrow M \longrightarrow H^0_*(\widetilde{M}) \longrightarrow \mathrm{Ext}^n_S(M, S(-n-1))^\vee \longrightarrow 0,$$

(b) For $i \ge 1$*,*

$$H^i_*(\widetilde{M}) \cong \mathrm{Ext}^{n-i}_S(M, S(-n-1))^\vee.$$

Corollary 6.6. *Let M be a graded S-module. Then for $i \geq 1$,*

$$h^i(\mathbb{P}^n, \widetilde{M}(d)) = \dim \operatorname{Ext}_S^{n-i}(M, S)_{-n-1-d},$$

and the dimension $h^0(\mathbb{P}^n, \widetilde{M}(d))$ of the space of global sections of $\widetilde{M}(d)$ is equal to

$$\dim M_d + \dim \operatorname{Ext}_S^n(M, S)_{-n-1-d} - \dim \operatorname{Ext}_S^{n+1}(M, S)_{-n-1-d}.$$

Proof of Theorem 6.5. Lemma 6.2 provides a proof of the theorem when $M = S$, and since all of our cohomologies commute with direct sums, it also proves the theorem when M is a graded free S-module. This proves the theorem in the situation when $\operatorname{pd}_S M = 0$.

Prove the general statement by induction on the projective dimension of the module M.

An alternate proof uses the two spectral sequences corresponding to the double complex $K^{p,q} = \mathcal{C}^p \widetilde{M} \otimes_S F_q$, where

$$0 \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is a graded free resolution of M. $\square$

6.2 Cohomology by approximation

The following result of Serre generalizes exercise 4.8.

Theorem 6.7. *Let M be a graded S-module, and i an integer. The for all $\ell \geq$ reg(M)-i,*

$$H^i_*(\widetilde{M})_{\geq 0} \cong \operatorname{Ext}^i_S(J_\ell, M)_{\geq 0},$$

where $J_\ell = (x_0^\ell, x_1^\ell, \ldots, x_n^\ell)$.

This may be proved either by using a spectral sequence, or by noting that the result holds for graded free modules, and then using induction on the projective dimension of M.

7 The Bernstein-Gelfand-Gelfand correspondence

We now describe a very cool relationship between the exterior algebra and cohomology of sheaves on $\mathbb{P}^n$. This section still needs to be written. For now, see [3, 1, 13] for details.

8 Project: The Hodge diamond

Exercise 8.1. For some varieties the approach above works fine, but for varieties with more complicated ideals, the computations can bog down. In this exercise/project you will address these concerns.

Given a smooth projective variety X (by its ideal), write a Macaulay2 routine to compute this $(d+1)\times(d+1)$ matrix of integers, by using as little computation as possible. Here are a few points to keep in mind:

- by duality, one doesn't need to compute all of these numbers,
- The Euler characteristic can be used to obtain one of the numbers from the rest (and usually this number isn't as hard to compute).
- If M and N are modules corresponding to the same sheaf, but the depth of M is 0, and the depth of N is at least 1, then the presentation and resolution of N are usually nicer.

8.1 Hodge diamond of the blowup of $\mathbb{P}^3$ along the twisted cubic curve

8.2

9 Example: A mystery curve

10 Example: A mystery surface

11 Example: Intersection numbers on a surface

12 Example: Rational surfaces and Castelnuovo's theorem

In this example, we use Castelnuovo's theorem to investigate the rationality of a specific smooth surface in $\mathbb{P}^4$. The specific surface was discovered by Decker, Popescu and Schreyer. First, let's recall Castelnuovo's theorem:

Theorem 12.1 (Castelnuovo)**.** *Let X be a smooth projective surface (over characteristic zero). X is rational if and only if $H^1(\mathcal{O}_X) = H^0(\omega_X^{\otimes 2}) = 0$.*

Construction of the surface

The following construction of $X = V(I) \subset \mathbb{P}^4$ looks like we are pulling a rabbit out of a hat. It is possible to motivate the construction using Beilinson monads, and while that is a very interesting story, we will concentrate on analyzing the surface via its ideal.

```
i135 : S = ZZ/32003[a..e];
```

```
i136 : F1 = S^{5}

        1
o136 = S

o136 : S-module, free, degrees {-5}

i137 : F2 = S^{3:4}

        3
o137 = S

o137 : S-module, free, degrees {-4, -4, -4}

i138 : G = S^{15:3}

        15
o138 = S

o138 : S-module, free, degrees {-3, -3, -3, -3, -3, -3, -3, -3, -3, -3 ···
```

The following produces a 4 by 18 matrix with entries random of degree 1 and 2, except for a single 3 by 3 block in the lower left.

```
i139 : M = matrix{{random(F1,F2++G)},
                 {map(F2,F2,0) | random(F2,G)}};

               4       18
o139 : Matrix S  <--- S

i140 : C = res coker M

        4      18       29       22       10       3
o140 = S  <-- S   <-- S   <-- S   <-- S   <-- S  <-- 0

       0      1       2       3       4       5      6

o140 : ChainComplex

i141 : betti C

o141 = total: 4 18 29 22 10 3
          -5: 1  3  3  1  . .
          -4: 3 15 26 15  . .
          -3: .  .  .  6 10 3
```

The kernel of the 29 by 16 submatrix of C.dd_3 corresponding to the first 16 columns is an ideal generated by 15 quintics and one sextic.

```
i142 : F = submatrix(C.dd_3,{0..15});

               29       16
o142 : Matrix S   <--- S

i143 : I = ideal syz F;

o143 : Ideal of S

i144 : betti res I

o144 = total: 1 16 29 18 4
           0: 1  .  .  . .
           1: .  .  .  . .
           2: .  .  .  . .
           3: .  .  .  . .
           4: . 15 26 15 3
           5: .  1  3  3 1
```

The surface $X = V(I) \subset \mathbb{P}^4$ is smooth of codimension 2, degree 9, and sectional genus 6.

```
i145 : (codim I, degree I)

o145 = (2, 9)

o145 : Sequence

i146 : genera I

o146 = {0, 6, 8}

o146 : List
```

The equations of I are messy, having been constructed using random polynomials, and so we don't display the specific equations defining X.

The cohomology of the surface

First we find $q = h^1(\mathcal{O}_X)$.

```
i147 : rank HH^1(sheaf(S^1/I))

o147 = 0
```

Alternatively, use local duality directly.

```
i148 : hilbertFunction({0},Ext^3(S^1/I,S^{-5}))

o148 = 0
```

The second plurigenus $P_2 = \dim H^0(\omega_X^{\otimes 2})$ is a bit more interesting.

```
i149 : KX = Ext^2(S^1/I,S^{-5});

i150 : KX2 = KX ** KX;

i151 : betti KX2

o151 = relations : total: 36 120
                       2: 36 120
```

Computing a free resolution of the module $K_X^{\otimes 2}$ is more difficult now, mainly due to the fact that the given presentation is not very efficient. We improve the presentation to depth 1 in the following way.

```
i152 : KX2sat = coker gens saturate(image presentation KX2);

i153 : betti res KX2sat

o153 = total: 36 75 63 30 6
           1:  . 15  .  .  .
           2: 36 60 63 30 6
```

Now we are in a position to easily compute P_2. By local duality, P_2 is the sum of the following two numbers, since Ext^5 is zero.

```
i154 : hilbertFunction({0},Ext^4(KX2sat, S^{-5}))

o154 = 0

i155 : hilbertFunction({0},KX2sat)

o155 = 0
```

Therefore $P_2 = 0$, so Castelnuovo's theorem implies that X is a rational surface.

Exercise 12.2. *Find a birational map between X and $\mathbb{P}^2$. X must be the blowdown at some points of the blowup of $\mathbb{P}^2$ at some points. Try to describe the surface along these lines.*

13 Example: Rational curves on a variety

14 Example: Rational connectivity

15 Example: Connectedness

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