Arizona Winter School 2005

Anabelian phenomena in arithmetic and geometry

— Aim/Description of the Course —

The aim of this course is to show that under certain arithmetic and geometry is encoded in Galois theory, provided some "anabelian hypothesis" is satisfied. The very first instance of this is the Theorem of Artin and Schreier (from the 1927) asserting that the absolute Galois group G_K of field K is finite non-trivial iff K is real closed; or equivalently $G_K \cong G_{\mathbf{R}}$. Thus the finiteness of a non-trivial absolute Galois group imposes very strong conditions on the elementary structure of the field in discussion. Finally, a modern "classic" in this direction is the theorem by Neukirch, Uchida (and Ikeda, Iwasawa) saying that two number fields K and L are isomorphic as fields iff their absolute Galois groups are isomorphic as profinite groups.

Plan for this course is as follows:

- Introduce the Galois theory and arithmetic fundamental group basics (but not explain in detail what was already done in the parallel courses, say by Matsumoto, etc.).

- Explain what are anabelian phenomena; in particular give Grothendieck's anabelian point of view and his anabelian conjectures.

- Present the state of the art with these conjectures, and emphasize some points from the proofs; say more about the so called Section conjecture.

- Discuss anabelian phenomena in the geometric situation, and illustrate this both in the curve case and in the birational situation. In particular recall the geometric case of a Conjecture of Shafarevich.

- Relation of the anabelian world with the Ihara/Oda–Matsumoto Conjecture concerning a geometric/combinatoric description of $G_{\mathbf{Q}}$, and relation with $\widehat{GT}$.

Projects (tentatively): 1) Tamagawa's trick for deducing (g, r) from the fundamental group of an affine curve. 2) The birational section conjecture over the p-adics. 3) Solvable groups as absolute Galois groups. 4) Deducing the anabelian conjecture for curves over number fields from the one over the p-adics (Mochizuki's Theorem). 5) Deducing the birational anabelian conjecture in the arithmetic case from the one in the geometric case.