Arizona Winter School lectures

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LECTURE 3

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3.1 The family

$$X : \quad t(X_0^4 + X_1^4 + X_2^4 + X_3^4) = X_0 X_1 X_2 X_3$$

$$f \downarrow$$

$$S = \mathbb{P}^1(\mathbb{C}) \setminus \{0, \tfrac{\text{4th roots of } 1}{4}\}$$

This is a family of K3-surfaces,

so

$$\dim_{\mathbb{Z}} H^2(X_t, \mathbb{Z}) = 22$$

There is a group acting

$$G = \underline{\{(\zeta_0, \zeta_1, \zeta_2, \zeta_3) \in \mu_4^4 \mid \prod \zeta_i = 1\}}$$

and we take $\qquad \Delta(\mu_4)$

$$V_{\mathbb{Z}} = \left(R^2 f_* \mathbb{Z}\right)_{prim}^{G}$$

as our PVHS.

<u>Proposition 3.1.1</u> The rank of V_Z is 3 and the Hodge numbers are
$$h^{2,0} = h^{1,1} = h^{0,2} = 1.$$

<u>Proof.</u> On the affine piece $X_0 \neq 0$ we have $X_t : f_t = 0$ with
$$f_t = t(1 + x_1^4 + x_2^4 + x_3^4) - x_1 x_2 x_3 .$$

Consider
$$\omega_t = \mathrm{Res}_{f_t = 0}\left(\frac{dx_1 \wedge dx_2 \wedge dx_3}{f_t}\right).$$

Check $\omega_t \in H^0(X_t, \Omega^3)$. [$S_4$-action].

Then clearly ω_t is G-invariant hence
$$h^{2,0}(V_Z) = 1. \left[\Rightarrow h^{0,2}(V_Z) = 1\right].$$

On the other hand (trick)
$$rk_Z \overset{H^2(X_t,\mathbb{Z})^G}{=} \frac{1}{\#G} \sum_{g \in G} \mathrm{Tr}\left(g \,|\, H^2(X_t)\right) \overset{LFP}{=} \frac{1}{4^2}\Big[\underbrace{22}_{g=1} + \underbrace{(4^2-1)2}_{g\neq 1}\Big] = 4$$

types: $(1,-1,1,-1) / (1,i,1,-i) / (1,i,i,-1)$ $\quad$ $72 - 2\cdot 15 + 22 = 42 + 22 = 64$

$3 \cdot 8 + 6 \cdot 4 + 6 \cdot 4 = 3 \cdot 24 = 72.$

But $\mathbb{Z}c_1(\mathcal{O}_{X_t}(1))$ invariant $\Rightarrow$ 3

Remark 3.1.2: The $(V_{\mathbb{Z}})^{\perp}$ is pure of type $(1,1)$!

3.2 PVHS of weight 2 rank 3 of type (1,1,1).

Suppose $(V_{\mathbb{Z}}, V_s^{p,q}, \psi)/S = D$ is such

Lemma 3.2.1 There is a $\mathbb{Q}$-basis e_1, e_2, e_3 such that

$$\psi = (2\pi i)^{-2}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \qquad \begin{array}{l} a \in \mathbb{Q}_{>0} \\ b, c \in \mathbb{Q}_{<0} \end{array}$$

Proof. Certainly we can diagonalize ψ. To see the signs:

$V_s^{1,1} = \overline{V_s^{1,1}}$ is real hence for $x \neq 0$

$x \in V_s^{1,1} \cap V_{s,\mathbb{R}}$ get $(2\pi i)^2 \psi(x,x) > 0$

$V_s^{2,0} \oplus V_s^{0,2}$ real and negative definite by (1.1.6.2). $\square$

In this basis, suppose that

(3.2.2) $$V_s^{2,0} = \mathbb{C}\begin{pmatrix}1\\ z_1\\ z_2\end{pmatrix}, \qquad z_i = z_i(s)$$

Then our conditions read

(1.1.4) z_i is hol. in D

(1.1.6.1) $$V_s^{1,1} \oplus V_s^{2,0} = \left(V_s^{2,0}\right)^{\perp}, \text{ hence}$$

$$(*) \quad a + bz_1^2 + cz_2^2 = 0, \text{ and}$$

~~$V_s^{1,1}$ is determined by $V_s^{2,0}$.~~

(1.1.6.2) $$(2\pi i)^2 \psi\left(i^{2-0}\begin{pmatrix}1\\ z_1\\ z_2\end{pmatrix}, \begin{pmatrix}1\\ \bar{z}_1\\ \bar{z}_2\end{pmatrix}\right) > 0,$$

i.e. $$-(a + bz_1\bar{z}_1 + cz_2\bar{z}_2) > 0$$

i.e. $$\boxed{-bz_1\bar{z}_1 - cz_2\bar{z}_2 > a} \quad (**)$$

and finally

$$(1.1.5)\quad \begin{pmatrix} 0 \\ \frac{dz_1}{ds} \\ \frac{dz_2}{ds} \end{pmatrix} \in V_s^{2,0} \oplus V_s^{1,1} = \left(V_s^{2,0}\right)^{\perp}$$

follows automatically since

$$\psi\left(\begin{pmatrix} 0 \\ \dot{z}_1 \\ \dot{z}_2 \end{pmatrix}, \begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix}\right) = \frac{1}{2}\frac{d}{ds}\,\psi\left(\begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix}\right)$$

$$= \frac{1}{2}\frac{d}{ds}\,0 = 0.$$

The "universal" variation lives over the period domain given by (*) & (**) which is an open disc. (again).

Remark 3.2.3 : Suppose $z_1, z_2 \in \mathbb{Q}(\sqrt{-d})$ $d \in \mathbb{Q}_{>0}$. Then $V_z^{2,0} \oplus V_z^{0,2}$ is a $\mathbb{Q}$-sub hodge structure and hence

$V_z^{1,1}$ lies over $\mathbb{Q}$ i.e the HS has ~~a~~ (1,1) Hodge-class. Conversely if V_z ~~has a Hodge class then $z_1, z_2 \in \overline{\mathbb{Q}}$.~~ (left to audience)

3.3 Picard Numbers

The *Picard number* is

$$\rho_t = \operatorname{rank}_{\mathbb{Z}} \frac{\text{divisors on } X_t}{\text{numerical equivalence}}$$

Proposition 3.3.1 There is a countable dense set of $t \in S = \mathbb{P}^1(\mathbb{C}) \setminus \{0, \frac{i^a}{4}\}$ such that $\rho_t = 20$ all others have $\rho_t = 19$.

Proof. For any $D \subset S$ we obtain $z_1(s), z_2(s)$ as in (3.2.2). We claim that these functions are not constant: ~~Use argument (of type (1.2.5.2) or (1.2.5.3))~~ ~~will see this~~ 1.2.5.2 or 1.2.5.3.
(monodr.) (inf. VHS)

Hence by Remarks 3.2.3 we get an extra $(1,1)$ Hodge class # for a countably dense set of $t \in D$.
By Rem 3.1.2 the rest is type $(1,1)$ already. Done by $(1,1)$ - theorem. $\square$

Remark 3.3.2 Jumpy t's in $\bar{\mathbb{Q}}$!

3.4 A miraculous formula

Stelling: $\rho_t = 20$ iff

$$j(t) = \frac{1}{4} \frac{1 - 3 \cdot 2^5 t^4 + \sqrt{1 - 2^8 t^4}}{1 - 2^7 + \sqrt{1 - 2^8 t^4}}$$

is the j-invariant of an elliptic curve having CM

Prop 3.4.1 Every HS $V_{t,\mathbb{Z}}$ is isom. as a $\mathbb{Q}$-HS to $\mathrm{Sym}^2(H^1(E))$ for some elliptic curve $E/\mathbb{C}$.

The idea is that starting with a HS V of wt 1 & rk 2 one can get

$$\mathrm{Sym}^2(V) \qquad \begin{array}{l} w=2 \\ rk=3 \\ \text{type } (1,1,1) \end{array}$$

Namely

$$\underset{V^{1,0}}{\begin{pmatrix} 1 \\ \tau \end{pmatrix}} \longmapsto \underset{V^{2,0}}{\begin{pmatrix} 1 \\ 2\tau \\ \tau^2 \end{pmatrix}}, \underset{V^{1,1}}{\begin{pmatrix} 1 \\ \tau+\bar{\tau} \\ \tau\bar{\tau} \end{pmatrix}}, \underset{V^{0,2}=\overline{V^{2,0}}}{\begin{pmatrix} 1 \\ 2\bar{\tau} \\ \bar{\tau}^2 \end{pmatrix}}$$

The intersection form on $\mathrm{Sym}^2(V)$ has matrix

$$\frac{1}{(2\pi i)^2}\begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

<u>Lemma</u> $\underline{3.4.2}$: $Q_0\psi$ as in 3.2.1 is equiv. to this if and only if $\exists\, \alpha \in V_{\mathbb{Q}}$ $\psi(\alpha,\alpha)=0$.

There is such a cycle:
Consider

$$X_t : f_t = t(1 + X_1^4 + X_2^4 + X_3^4) - X_1 X_2 X_3 = 0$$

~~for $|t|$ small and let~~

$$\sigma_t : S^1 \times S^1 \longrightarrow X_t$$

$$(\phi_1, \phi_2) \longmapsto X_1 = |t|^{1/3} e^{2\pi i \phi_1}$$

$$X_2 = |t|^{1/3} e^{2\pi i \phi_2}$$

$X_3 =$ unique complex number close to $\frac{t}{|t|^{2/3}} \cdot e^{2\pi i(-\phi_1 - \phi_2)}$

s.t. $f_t(X_1, X_2, X_3) = 0$.

Set

$$\alpha = \sigma_{t*}([S^1 \times S^1]) \in H_2(X_t, \mathbb{Z})$$

then

$$\begin{cases} \alpha \frown \alpha = 0 & \text{in } H_0 \\ \alpha \in H_2(X_t, \mathbb{Z})^G & \\ \alpha \neq 0 & \end{cases}$$

$\alpha \cup \alpha = 0$, explicit computation; just let $X_1 = (1+\varepsilon)|t|^{1/3} e^{2\pi i \phi_1}$ [move α to α', $\alpha \cap \alpha' = 0$

α is fixed by G: α = vanishing cycle for $(1:0:0:0) \in X_0$ and $(1:0:0:0)$ is fixed.

$\alpha \neq 0$. We compute explicitly

$$\int_\alpha \omega_t = \pm(2\pi i)^2 + \underline{\underline{\text{small}}}$$

Namely

$$\omega_t = \operatorname{Res}\left(\frac{dx_1 \wedge dx_2 \wedge dx_3}{f_t}\right)$$

$$= \operatorname{Res}\left(\frac{dx_1 \wedge dx_2}{\frac{\partial f_t}{\partial x_3}} \wedge \frac{df_t}{f_t}\right)$$

$$= \left.\frac{dx_1 \wedge dx_2}{-x_1 x_2 + 4t x_3^3}\right|_{X_t}$$

pull back by σ_t to get $(2\pi i)^2 \, d\phi_1 \wedge d\phi_2 + \underline{\underline{\text{small}}}$.