Arizona Winter School lectures

By Johan de Jong

LECTURE 1

Lecture 1

1.1 Polarized Variations of Hodge Structures
PVHS's

Let S be a complex (analytic) manifold

eg. $S = \quad D = \{z \in \mathbb{C},\ |z| < 1\}$

$D^* = D \setminus \{0\}$

D^n (poly disc)

$\mathbb{P}^n(\mathbb{C}) \setminus$ (divisor)

Set

$\mathbb{Z}(k) = (2\pi i)^k \mathbb{Z}$ Hodge structure of type $(-k,-k)$ and weight $-2k$

will also denote the constant sheaf with value $\mathbb{Z}(k)$.

Suppose we are given data

(1.1.1) $V_{\mathbb{Z}}$ a local system of ~~ab~~ free abelian groups over S (i.e. locally constant sheaf with fibre $\mathbb{Z}^r$ some r)

(1.1.2) $\psi : V_{\mathbb{Z}} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} \longrightarrow \mathbb{Z}(-w)$

a bilinear pairing, symmetric if w even, alternating if w odd

(1.1.3) For each point $s \in S$ a Hodge structure of weight w on $V_{s,\mathbb{Z}}$ = fibre of $V_{\mathbb{Z}}$ at s :

$$V_{s,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{\substack{p+q=w \\ p,q \geq 0}} V_s^{p,q}$$

<u>Definition</u>: Data as above define a <u>PVHS's</u> iff the following conditions are satisfied

(1.1.4) (<u>holomorphicity</u>) Setting
$$F_s^p = \bigoplus_{p' \geq p} V_s^{p',q'}$$
the filtration
$$0 = F_s^w \subset F_s^{w-1} \subset \dots \subset F^0 = V_{s,\mathbb{C}}$$
depends analytically on $s \in S$: i.e. $\exists$

locally direct summands

$$\mathcal{F}^p \subset V_{\mathcal{O}} \overset{\text{def}}{=} V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_S$$

such that $F_s^p = \mathcal{F}_s^p$.

(1.1.5) (Griffiths Transversality) the canonical connection

$$\nabla : V_{\mathcal{O}} \longrightarrow \Omega^1_S \otimes_{\mathcal{O}} V_{\mathcal{O}}$$

~~set~~ and $\mathcal{F}^{\bullet}$ satisfy

$$\nabla(\mathcal{F}^p) \subset \Omega^1_S \otimes_{\mathcal{O}} \mathcal{F}^{p-1}.$$

(1.1.6) (Polarization) For every $s \in S$ the map ψ_s defines a polarization of the Hodge structure on $V_{s,\mathbb{Z}}$:

(1.1.6.1) $\psi_s : V_{s,\mathbb{Z}} \otimes_{\mathbb{Z}} V_{s,\mathbb{Z}} \longrightarrow \mathbb{Z}(-w)$

is a morphism of Hodge structures

$\left[\psi_s(V_s^{p,q} \otimes V_s^{p',q'}) = (0) \text{ unless } p+p' = q+q' = w\right]$

(1.1.6.2) $(2\pi i)^w \psi(i^{p-q} x, \bar{x}) > 0$ for $0 \neq x \in V_s^{p,q}$.

<u>Example</u> (1.1.7) Assume $w=1$, $V_{\mathbb{Z}}=\mathbb{Z}^2$ is constant and

$$\psi : V_{\mathbb{Z}}\otimes V_{\mathbb{Z}} \longrightarrow \frac{1}{2\pi i}\mathbb{Z},$$

$$\psi\left(\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}c\\d\end{pmatrix}\right)=\frac{1}{2\pi i}(ad-bc).$$

Then in any point $s\in S$ we have a weight 1 Hodge structure of rank 2. Hence $\dim V_s^{1,0}=\dim V_s^{0,1}=1$, and

$$F_s^1=V_s^{1,0}=\left\langle\begin{pmatrix}1\\\tau(s)\end{pmatrix}\right\rangle, \quad V_s^{0,1}=\left\langle\begin{pmatrix}1\\\overline{\tau(s)}\end{pmatrix}\right\rangle$$

for some $\tau(s)\in\mathbb{C}\setminus\{0\}$ [~~left to the reader~~]

(1.1.4): $\tau(s)$ is a holom. function

(1.1.5): no condition since $F^0=V_0$

(1.1.6.1): $\psi\left(\begin{pmatrix}1\\\tau\end{pmatrix},\begin{pmatrix}1\\\tau\end{pmatrix}\right)=0$ ✓

(1.1.6.2): $(2\pi i)\psi\left(i\begin{pmatrix}1\\\tau\end{pmatrix},\begin{pmatrix}1\\\bar\tau\end{pmatrix}\right)=2\pi i\cdot\frac{1}{2\pi i}(i\bar\tau-i\tau)$

$$=2\,\mathrm{Im}(\tau)>0$$

So $\mathrm{Im}(\tau(s))>0$.

Conclusion: In ~~the~~ case $w=1$, $V_{\mathbb{Z}} = \mathbb{Z}^2$ and ψ as above, the "universal" PVHS lies over

$$\mathfrak{h} = \{ \tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0 \}.$$

Any other (with trivial monodromy) is obtained by pulling back via a (holomorphic) "period" map

$$S \longrightarrow \mathfrak{h}$$

1.2 PVHS and algebraic geometry

Let $f : X \to S$ be a smooth projective morphism of algebraic varieties over $\mathbb{C}$, with connected fibres of dimension, and assume S non-singular as well.

Consider

$$V_{\mathbb{Z}} = R^w f_*(\mathbb{Z}) / \text{torsion} .$$

Since f is topologically a fibration, this is a local system on S, and

$$V_{s,\mathbb{Z}} = H^w(X_s, \mathbb{Z}) / \text{torsion}$$

The Hodge structures: $V_s^{p,q} = H^q(X_s, \Omega^p)$ where we use that $H^w(X_s,\mathbb{Z})\otimes\mathbb{C} = H^w(X_s,\mathbb{C}) = H^w(X_s, \Omega^{\cdot}_{X_s})$.
The cup product gives a map

$$V_{\mathbb{Z}} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} \xrightarrow{\cup} R^{2w} f_* \mathbb{Z} \underset{can}{\overset{\int}{\cong}} \mathbb{Z}$$

$$x \otimes y \longmapsto \int x \cup y$$

and we set

$$\psi(x,y) := (-1)^{\frac{w(w-1)}{2}} \cdot \frac{1}{(2\pi i)^w} \int x \cup y$$

All axioms are satisfied except for possibly (1.1.6.2). To get this we take a relatively ample sheaf $\mathcal{L}$ on X and set

$$V_{\mathbb{Z}} \supset V'_{\mathbb{Z}} = \operatorname{Ker}\left(R^w f_* \mathbb{Z} \xrightarrow{\cup c_1(\mathcal{L})} R^{w+2} f_* \mathbb{Z}\right)$$

with induced Hodge structures.

<u>Theorem</u> (4.2.1): $(V'_{\mathbb{Z}}, V_s'^{p,q}, \psi|_{V'})$ is a PVHS.

<u>Example 1.2.2</u> Consider the family of curves

$$X : \quad t\left(X_0^3+X_1^3+X_2^3\right) = X_0X_1X_2$$

over $S=\mathbb{P}^1(\mathbb{C}) \setminus \left\{ \tfrac{1}{3}, \tfrac{e^{2\pi i/3}}{3}, \tfrac{e^{4\pi i/3}}{3}, 0 \right\}$.

For each $t \in S$ the curve X_t is a nonsingular projective curve of genus 1. Hence we obtain a PVHS $(V_{\mathbb{Z}}, V_{\mathbb{C}}^{p,q}, \psi)$ over S. For any $D \overset{i}{\hookrightarrow} S$ we obtain by pull back a PVHS over D as in example 1.1.7 (since we know ψ_t is unimodular $\forall\, t \in S$) and hence

$$\begin{tikzcd} D \arrow[r, "\tau"] \arrow[d, hook, "i"'] & \mathfrak{h} \\ S & \end{tikzcd}$$

τ as in example 1.1.7

<u>Claim 1.2.3:</u> τ is not constant.

Before we prove the claim let us observe the consequence:

Consequence $1.2.\cancel{3}4$ For countably many values of t the curve X_t has CM (ie. corresponds to a $\tau \in \mathfrak{h}$ which is imaginary quadratic).
Reason: such points are dense in $\mathfrak{h}$.

1.2.5 ~~Three~~ Proofs of the claim (1.2.3)

(I) (Torelli) $(1.2.\cancel{3}5.1)$

τ constant $\Longrightarrow$ Isom. class of $\overbrace{H^1(X_t,\mathbb{Z})}^{\text{Hodge structure}}$ indep. $t \in D$

$\Longrightarrow$ " " " $J(X_t)$ " " "

$\overset{\text{Torelli}}{\Longrightarrow}$ " " " X_t " " "

If $\infty \in D$ then False as $X_\infty \not\cong X_{t}$ (eg compute j-invariant) or autom group)

(II) (Monodromy) $(1.2.\cancel{3}5.2)$

τ constant $\Longrightarrow$ F^1 does not move over D

$\Longrightarrow \nabla(F^1) \subset \Omega^1 \otimes F^1$ over D

$\overset{\text{anal. cont}}{\Longrightarrow} \nabla(F^1) \subset \Omega^1 \otimes F^1$ over S

$\Longrightarrow V_{\mathbb{C}}$ has a rank 1 local subsystem over S, namely $(F^1)^{\nabla}$

$\Longrightarrow$ The monodromy representation
$\rho : \pi_1(S, t_0) \longrightarrow \mathrm{Aut}(V_{t_0,\mathbb{Z}})$.

preserves $V_{t_0}^{1,0} \subset V_{t_0,\mathbb{C}}$

$\xrightarrow[\text{Lemma}]{\text{Easy}}$ ~~$\mathrm{Im}(\rho)$ does not have a nontrivial unipotent element~~

False : Look at ~~the~~ $\rho(\text{loop around } t=0) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

(1.2.5.3) (III) (Infinitesimal) ~~for~~ Griffiths transversality says $\nabla(\mathcal{F}^p) \subset \Omega^1_S \otimes \mathcal{F}^{p-1}$ and this gives linear maps

$$F_s^p / F_s^{p+1} \longrightarrow \Omega^1_{S,s} \otimes F_s^{p-1} / F_s^p ,$$

or

$$T_{S,s} \otimes V_s^{p,q} \longrightarrow V_s^{p-1,q+1} .$$

If the variation is "constant" then these are zero. In the set up of 1.2 this map is equal to the composition

$$T_{S,s} \otimes H^q(X_s, \Omega^p) \xrightarrow{\kappa \otimes \mathrm{id}} H^1(X_s, T_{X_s}) \otimes H^q(X_s, \Omega^p)$$
$$\downarrow \cup$$
$$H^{q+1}(X_s, \Omega^{p-1})$$

Reference: M. Green, Inf. Methods in Hodge Theory LNM 1594 where

$$\kappa : \cancel{T}\, T_{S,s} \longrightarrow H^1(X_s, T_{X_s})$$

is the Kodaira Spencer map. A computation shows this to be nonzero in this case.

Remark $\boxed{1.2.6}$ In case of a ~~Lefschetz~~ pencil

$$X_t : \quad tF = G \quad , \quad F, G \in \Gamma(P, \mathcal{L})$$

inside P we get

- $$0 \to T_{X_{t_0}} \longrightarrow T_P\big|_{X_{t_0}} \longrightarrow N_{X_{t_0}} \to 0$$
- $$N_{X_{t_0}} \cong \mathcal{L}\big|_{X_{t_0}}$$

and

$$\kappa\left(\frac{\partial}{\partial t}\right) = \text{Image of } F \text{ via boundary } H^0(N_{X_{t_0}}, P) \to H^1(T_{X_{t_0}}).$$

Reference P. Deligne, Local behavior of Hodge structures at infinity.