A note on "risk-neutral valuation"

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This note is meant to be a short justification of the principle that options can be valued by assuming that the underlying stock grows at the risk-free rate.

A binomial model. Consider a stock worth S at time 0 (now). We assume that at time T it will be have gone up to Su or down to Sd. (We have no information about the probabilities of the two outcomes.) Note that if r is the "risk-free" rate of return, then the sensible values of u and d are those where $u \geq e^{rT} \geq d$. (And so "down" is a relative term.)

Consider a derivative whose worth at time T will be D_u if the stock goes up, and D_d if it goes down. Our first goal is to find D, the current fair value of the derivative. (The argument to follow is quite general, but if you haven't thought much about this, or are uncomfortable with "shorting" then you can think of the derivative as being a European call option.) To find D, consider buying Δ shares of the stock and selling one derivative. The net cost is $\Delta S - D$. (We're assuming we can buy fractional shares, and, in the general case, short stock. We're also ignoring trading costs.) At time T we can sell the shares and buy back the derivative for a net total of $\Delta Su - D_u$ (if the stock goes up) or $\Delta Sd - D_d$ (if it goes down). If we let $\Delta = \frac{D_u - D_d}{S(u-d)}$, then simple algebra shows that the value of our portfolio at time T is independent of whether the stock goes up or down:

$$\Delta Su - D_u = \Delta Sd - D_d.$$

Thus, by definition, our portfolio is risk-free, so has to grow at the risk-free rate. That means its current value is

$$\Delta S - D = e^{-rT}(\Delta Su - D_u) = e^{-rT}(\Delta Sd - D_d).$$

This equation can then be solved for D. Note that we have found the value of D without knowing anything about the likelihood that the stock will go up or down.

Expected value interpretation. With a little algebra, the last displayed equation is equivalent to

$$D = e^{-rT}\left(pD_u + (1-p)D_d\right) \qquad (*)$$

where $p = (e^{rT} - d)/(u-d)$. Note that for sensible u and d we have $0 \leq p \leq 1$. It's natural to interpret p as the probability that the stock goes up, in which case the current value of the derivative is simply its expected future value, discounted to the present. What does this interpretation say about the expected future value of the stock? We have

$$pSu + (1-p)Sd = \frac{e^{rT} - d}{u-d}Su + \frac{u - e^{rT}}{u-d}Sd = e^{rT}S. \qquad (**)$$

In other words, if p is the probability that the stock goes up, then the future value of the stock is growing at the risk-free rate!

Comment. We are *not* saying that stocks grow at the risk-free rate, nor that p is the probability that the stock will go up. What we are saying is that if we let p be the unique number that makes $(**)$ hold, view p as a probability, and then value the option according to the naive model $(*)$, then we get the correct answer.

Terminology. It is widely believed that investors require extra return as compensation for risk (essentially defined to be variability of returns). If investors were indifferent to risk ("risk-neutral") then they would only be concerned with expected future values (not variances) and by arbitrage arguments, all securities would have the same expected return, namely the risk-free rate. Thus the procedure of assuming a stock's future value grows at the risk-free rate and computing the value of a derivative as a discounted future value is sometimes called "risk-neutral valuation." The preceding discussion shows that, at least with a very simple model, this gives the correct answer.

Generalizations and a reference. Similar arguments can be made in more complicated contexts, e.g., where stock prices are modeled by Brownian motion in continuous time (the context of the Black-Scholes formula). See John C, Hull, "Options, Futures, and Other Derivatives" (Prentice Hall, 1999) for much more on the subject.