Project for "p-adic modular forms" lectures.

KMB, 6 Feb 2001

In this project I will sketch a method which I believe will give an explicit formula for all the entries of a matrix representing U on overconvergent 2-adic modular forms of level 1 and any even weight k. This explicit formula will make it much easier to say things about the overconvergent slopes in these cases—for example it should lead very easily to explicit lower bounds for Newton polygons. Even so, there are still plenty of conjectures, which will give us lots of things to think about, even assuming we can get the explicit formulae to come out.

The genesis of the project is Lawren Smithline's PhD thesis. Before I read this thesis, I had no idea that it was possible to compute explicit matrices representing U_p on the space of weight k overconvergent modular forms. Now I've seen the idea, I realise that in fact it is not too difficult. Smithline was concerned with the case $p = 3$ and tame level 1, and weight 0. Using an extension and minor simplification of his ideas, this project will attempt to deal with the case $p = 2$, tame level 1, and any even integer weight.

Introduction and weight 0

There is a unique Eisenstein series $E_2(q) = 1 + 24q + \ldots$ of weight 2 and level 2. It is explicitly given as $1+24\sum_{n\geq 1} a_n q^n$, where a_n is the sum of the odd divisors of n. Unfortunately (at least for those of you who do not like computers) it may occasionally be necessary to have to compute the q-expansions of some other forms too. But there are programs to do this sort of thing, and assuming I've brought my laptop to the conference, I will have access to them, so I could do any computations required.

Using standard facts about p-adic modular forms, check that $E_2(q)$ and $E_2(q^2)$ are both overconvergent 2-adic modular forms of level 1 and weight 2. Check furthermore that $E_2(q^2)$ has no zeros on the ordinary locus. Hence the ratio $t = E_2(q)/E_2(q^2)$ is a p-adic modular form of weight 0. It will be an r-overconvergent form for some $r < 1$—can you say anything about r? This is not necessary, but it would be nice. I think it might be possible to compute some explicit bounds but I'm not sure. It might involve some tricks with the j-invariant.

Now $t - 1$ can be thought of as an overconvergent function on some strict neighbourhood X of the ordinary locus at level 1. Why will t have a unique zero on X (assuming X is a small enough strict neighbourhood)? The zero is clearly at the cusp. The region $|t - 1| \leq B$ will define a disk D in X, for appropriate B. Let d be an appropriate multiple of $t - 1$, so that the disk $|d| \leq 1$ does indeed define a strict neighbourhood of the ordinary locus, which is preserved by the U operator. You probably have a range of choices for the normalisation actually, each of which will give slightly different discs, but the theory should be independent of these choices. Can you explicitly say, given your choices, that functions on D are r-overconvergent functions for some explicit $r < 1$?

Now for the fun part. An explicit basis for modular forms on D is just $\{1, d, d^2, d^3, \ldots\}$. What is $U(d^n)$ in terms of this basis? A computer might be helpful, but in fact the actual amount of computer calculations you have to do is minimal if you see all the tricks. Now one can see the matrix for U in weight 0.

Warm-up question: can you get sufficiently good bounds on the elements of this matrix to deduce a quadratic lower bound for the Newton polygon of the characteristic power series of U in weight 0? Smithline indicates how to do this in his thesis, although he uses a different choice of d.

Interlude: a conjecture about weight 0 slopes.

I conjecture that the valuations of the eigenvalues of U on 2-adic cuspidal overconvergent forms of weight 0 are 3,7,13,15,17,.... I will explain how to generate this sequence. First of all start with the sequence $2, ?, ?, 2, 2, ?, ?, 2, 2, ?, ?, 2, \ldots$, where the ?s are unknown entries that we will work out in due course. This sequence could be described as "2, then alternate double-?s and double-2s".

Now fill in the question marks, using the sequence "4, and then alternate double-?s and double 4s", and simply skipping over the terms we know already. We get the sequence $2, 4, ?, 2, 2, ?, 4, 2, 2, 4, ?, 2, 2, ?, 4, 2, \ldots$.

Now do the same thing with 6: fill in the remaining question marks with the sequence "6, and then alternate double-?s and double-6s".

Eventually the sequence goes $2, 4, 6, 2, 2, 8, 4, 2, 2, 4, 10, 2, \ldots$. Now form the sequence $1, 3, 7, 13, 15, 17, 25, \ldots$ whose successive differences are the terms in our $2, 4, 6, 2 \ldots$ sequence. Throw away the first 1 and you get what I firmly believe should be the cuspidal slopes at weight 0. Can you prove that this is the case?

Other weights.

Now we have done all this work in weight 0, it is surprisingly easy to work out a matrix for U at weight $k = 2m$ for all integers m. Firstly you have to check that you can use $E_2(q)^m d^n$ for $n = 0, 1, 2, \ldots$ as a basis. Then you have to work out $U(E_2(q)^m d^n)$ in terms of this basis. Hence you can work out a matrix representing U in weight $2m$. Again the warm-up question: can you work out an explicit quadratic lower bound for the Newton polygon of the characteristic power series?

I have conjectures for the valuations of the eigenvalues of U for arbitrary even weights. They are elementary, but messy, to explain and I won't do it here (bug me for a preprint). But here are some consequences of these conjectures, all of which are, I believe, open. I will ignore the "Eisenstein slope", the 0 which occurs at every weight, here, so "slope" below refers to the slopes of the overconvergent cusp forms, a “formula” for which you have computed above. So here are the conjectures.

Lots of conjectures.

I have no idea how to do any of these. Do the calculations you have so far help?

- All slopes at all even weights are integers.
- At weight -2 all slopes occur with multiplicity precisely 2. Similarly at weight 4—except for the first slope, which is 3, all other slopes occur with multiplicity 2. These weights are related by the theta operator, so the two observations are basically equivalent.
- At weight 0 and 4 all slopes are odd. At weights 2 and -2 all slopes are even (again some of these things imply others).
- The nth slope is between $3n$ and $6n$, and furthermore these bounds are obtained infinitely often. In particular there is a quadratic upper bound for the Newton polygon.
- At weight $k > 0$ there are no forms of slope s for s in the range $(k/3, 2k/3)$, other than forms of slope $(k-2)/2$ (these bounds might be slightly wrong, but I think they're OK. The idea that there might be a hole here is basically due to Gouvea.)
- The Gouvea-Mazur conjectures: if k and k' are congruent mod 2^n then the number of times a slope $s < n$ shows up at weights k and k' should be the same. This is basically known if one replaces 2^n by something like 2^{n^2}, but is still open in the form above.