P-adic Modular Forms by Kevin Buzzard
Lecture 2

• We know modular forms exist because over $\mathbf{C}$ there exists an explicit example (according to the classical definition), namely the Eisenstein series: $E_k : \tau \mapsto \sum_{\substack{m,n\in\mathbf{Z} \\ \text{not both } 0}} \frac{1}{(m\tau+n)^k}, k \geq 4$. This is a level 1 modular form of weight k. A known fact is that $E_{p-1} \equiv 1 \pmod{p}$, where p is prime and $p \geq 5$. So in the ring $\mathbf{Z}_p[[q]]$, E_{p-1} "looks" invertible becuase it has a power series where the constant term is a unit. However, no classical forms are of negative weight so $1/E_{p-1}$ is not a classical form. In addition, for example, $E_4(\pi i/3) = E_4(i) = 0$ so $1/E_4$ is not a holomorphic function on the upper half plane.

• The following is the Deligne/Katz approach to defining a p-adic modular form.

We have E/R, where E is an elliptic curve over R, an $\mathbf{F}_p$-algebra. Now let $\omega \in H^0(E, \Omega^1_{E/R})$ and $\eta \in H^1(E, \mathcal{O}_E)$ be its dual. Consider the absolute Frobenius map, $F_{\text{abs}} : \mathcal{O}_E \to \mathcal{O}_E,\ f \mapsto f^p$, an additive homomorphism of sheaves of abelian groups. Now define $A(E/R, \omega) \in R$, (which is actually the Hasse invariant) by setting $F^*_{\text{abs}}(\eta) = A(E/R, \omega) \cdot \eta$ which gives us that $A(E/R, \lambda\omega) = \lambda^{1-p} A(E/R, \omega),\ \ \lambda \in R^\times$.

So A is a modular form of weight $p-1$. Note that the boundedness condition for a modular form is satisfied by looking at $A(\text{Tate}(q), \omega_{\text{can}})$ since the restriction of a plane curve over $\mathbf{F}_p[[q]]$ is the Tate curve over $\mathbf{F}_p((q))$. Now $A(\text{Tate}(q), \omega_{\text{can}}) = 1$ so if $p \geq 5$, then $E_{p-1} \equiv A \pmod{p}$. Therefore $A = E_{p-1} \pmod{p}$ by the q-expansion principle, which says that two modular forms of level 1 and the same weight are equal if they have the same q-expansion.

We want a p-adic theory of modular forms that strongly identifies a modular form with its q-expansion so that what "looks" invertible, as in $E_{p-1} \pmod{p}$, is invertible. So because the Hasse invariant $A(E/R, \omega) = 0$ if and only if E is supersingular, we want to somehow throw away elliptic curves which are supersingular or have supersingular reduction.

- Katz's definition of a p-adic modular form:

Let $p \geq 5$, R_0 be the ring of integers in a finite extension of $\mathbf{Q}_p$, and R an R_0-algebra in which p is nilpotent.
A test object is

1. an elliptic curve E/R
2. a nowhere-vanishing differential $\omega \in H^0(E, \Omega^1_{E/R})$
3. an element $Y \in R$ such that $Y \cdot E_{p-1}(E/R, \omega) = 1 \in R$.

A *p-adic modular form* of level 1 of weight k defined over R_0 is a rule f sending $(E/R, \omega, Y)$ to an element of R such that

a) $f(E/R, \lambda\omega, \lambda^{p-1}Y) = \lambda^{-k} f(E/R, \omega, Y)$

b) $f(\text{Tate curve over } R_0/p^n R_0((q)), \omega_{\text{can}}, 1)$ is in $R_0/p^n R_0[[q]]$ for all $n \geq 1$.

and f depends only the isomorphism class of data and behaves well under pullback.

Note that classical modular forms over R_0 are already p-adic modular forms.

- The class of p-adic modular forms is too large to work with, but we can create subtlety through the following definition:

Let R_0 be the ring of integers in a finite extension of $\mathbf{Q}_p$ and choose $\rho \in R_0 \backslash 0$, and R an R_0-algebra in which p is nilpotent.
A ρ-overconvergent test object is

1. an elliptic curve E/R
2. a nowhere-vanishing differential $\omega \in H^0(E, \Omega^1_{E/R})$
3. an element $Y \in R$ such that $Y \cdot E_{p-1}(E/R, \omega) = \rho$.

A *ρ-overconvergent modular form* is a rule on these test objects satisfying the conditions (a) and (b) in the definition of a p-adic modular form.

Note that if $|\rho| < 1$ then some of these test objects might have supersingular geometric fibers.

- We don't want to throw away too much so the idea is to define E, an elliptic curve, as having "very supersingular reduction" if $A(E(\text{mod } p) \ / \ R/pR) = 0 \in R/pR$. Now if R is the ring of integers in a highly ramified extension of $\mathbf{Q}_p$ then R/pR could be huge so maybe there are a lot of elliptic curve whose reductions are supersingular but not very supersingular.