PAWS 2026: ALGEBRAIC GEOMETRY VIA COMPUTATION
PROBLEM SET 5

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Question 1: One Conic, Two Possible Limits. Work in $\mathbb{P}^2$ with coordinates $[x : y : z]$.

(a) Assume $t \in \mathbb{K}^\times$. Show that $xy - tz^2$ and $x^2 - tyz$ define smooth conics over an algebraic closure of $\mathbb{K}$. You may use the fact that a reducible plane conic is singular to conclude that they are irreducible.

(b) For the two families displayed above, set $t = 0$ and describe the zero sets. Prove

$$\langle xy \rangle = \langle x \rangle \cap \langle y \rangle, \qquad \sqrt{\langle x^2 \rangle} = \langle x \rangle.$$

Which limit is reduced? Which is irreducible as a point set but not as an ideal with multiplicity remembered?

(c) Fix lex order $x > y > z$ and regard t as a coefficient. The two divisors below are monic, so division never asks us to invert a nonunit of $\mathbb{K}[t]$ and the familiar argument works verbatim. Show that $xy - tz^2$ is monic with leading monomial xy. Use division to prove that the monomials not divisible by xy form a $\mathbb{K}[t]$-basis of

$$\mathbb{K}[t, x, y, z]/\langle xy - tz^2 \rangle.$$

Do the same for $x^2 - tyz$ and the monomials not divisible by x^2.

(d) Take the degree-d part in x, y, z. Explain why both modules in part (c) are free $\mathbb{K}[t]$-modules of rank $2d + 1$. After setting $t = a$, why does every fiber have Hilbert function $2d + 1$?

(e) What geometric properties changed at $t = 0$? Which numerical information did not change?

Question 2: Reading Geometry from the Polynomial.

(a) Since $H_{\mathbb{P}^n}(d) = \binom{n+d}{n}$, show that projective n-space has dimension n and degree 1.

(b) Show that a hypersurface in $\mathbb{P}^n$ has dimension $n - 1$ and degree e.

(c) A conic has Hilbert polynomial $2d + 1$. Read from this expression its dimension, degree, and arithmetic genus. Why do the smooth conic, the pair of lines, and the double line have the same answers?

(d) Let $X = \{p_1, \ldots, p_r\} \subseteq \mathbb{P}^n$ be a set of distinct $\mathbb{K}$-rational points, and choose a nonzero homogeneous-coordinate representative for each p_i. Evaluation at these representatives gives a linear map $S_d \to \mathbb{K}^r$; changing a representative only rescales the corresponding coordinate. For each $i \neq j$, choose a linear form ℓ_{ij} which vanishes at p_j but not at p_i. Show that

$$L_i = \prod_{j \neq i} \ell_{ij}$$

vanishes at every p_j except p_i.

(e) Choose a linear form m_i which is nonzero at p_i. For every $d \geq r - 1$, multiply L_i by m_i^{d-r+1} and rescale it to take the value 1 at p_i. Why do the resulting degree-d forms behave like homogeneous Lagrange polynomials?

(f) Use these forms to prove that $S_d \to \mathbb{K}^r$ is surjective for $d \gg 0$. Identify its kernel and conclude that $P_{S/\mathbb{I}_{\mathbb{P}^n}(X)}(d) = r$.

(g) Why would replacing the ideal of r reduced points by a nonradical ideal supported at those points preserve the visible set but potentially increase the eventual constant Hilbert polynomial?

Question 3: The Hilbert Function of the Twisted Cubic. Assume $\mathbb{K}$ is infinite, so Lecture 3 identifies I_C with the homogeneous vanishing ideal of $\nu_3(\mathbb{P}^1)$. The algebraic kernel calculation, and hence the Hilbert-function computation below, remains valid over every field.

(a) The substitution $w \mapsto s^3$, $x \mapsto s^2u$, $y \mapsto su^2$, $z \mapsto u^3$ sends a degree-d monomial in w, x, y, z to a degree-$3d$ monomial in s, u. Show that every monomial $s^{3d-k}u^k$, for $0 \le k \le 3d$, occurs in this way. Hint: write k as a sum of d integers chosen from $\{0, 1, 2, 3\}$.

(b) Deduce that the degree-d substitution map

$$(S/I_C)_d \longrightarrow \mathbb{K}[s, u]_{3d}$$

is an isomorphism.

(c) Conclude that

$$H_{S/I_C}(d) = 3d + 1 \qquad (d \ge 0).$$

Read off the dimension, degree, and arithmetic genus of the projective twisted cubic.

(d) Any homogeneous initial ideal of I_C must have exactly how many degree-d standard monomials? This is the numerical target against which we will check the special fiber below.

Question 4: The Twisted Cubic Becomes a Chain of Three Lines. Refine the weight order by any monomial order to break ties.

(a) Verify the three weight comparisons above. Show that the S-polynomials of the pairs (g_1, g_2) and (g_2, g_3) reduce to zero using the identities

$$zg_1 - yg_2 = -xg_3, \qquad xg_2 - wg_3 = yg_1.$$

(b) The leading monomials wy and xz are relatively prime, but let us check their pair explicitly. Verify

$$xzg_1 - wyg_3 = -x^2g_3 + y^2g_1.$$

Explain why every leading product on the right has weight strictly smaller than the common leading monomial $wxyz$ on the left. Conclude from Buchberger's criterion that $G = (g_1, g_2, g_3)$ is a Gröbner basis for the weight-refined order and that

$$\mathrm{in}_\omega(I_C) = \langle wy, wz, xz \rangle.$$

(c) Now ask *Macaulay2* to make the same weight computation. Predict both Boolean outputs before running the code.

```
restart
Rw = QQ[w,x,y,z,
        MonomialOrder=>{Weights=>{3,1,0,0},GRevLex}];

ICw = ideal(w*y-x^2,w*z-x*y,x*z-y^2);

weightInitial = leadTerm(1,ICw)
termInitial   = leadTerm ICw

weightInitial == ideal(w*y,w*z,x*z)
termInitial == weightInitial
```

The command `leadTerm(1,ICw)` stops after the weight comparison and retains the entire top-weight part. The command `leadTerm ICw` also uses graded reverse lexicographic order to break ties. Why do they agree here? What could change if one of the relevant comparisons were a tie?

(d) Prove the following intersection of prime ideals:

$$M = \langle y, z\rangle \cap \langle w, z\rangle \cap \langle w, x\rangle.$$

Hence $\mathbb{V}_{\mathbb{P}^3}(M)$ is the union of the coordinate lines

$$L_{wx} = \mathbb{V}(y, z), \qquad L_{xy} = \mathbb{V}(w, z), \qquad L_{yz} = \mathbb{V}(w, x).$$

Show that the first two meet at $[0 : 1 : 0 : 0]$, the last two meet at $[0 : 0 : 1 : 0]$, and the first and last are disjoint. Draw the resulting chain.

(e) For $d \geq 1$, a standard monomial of M uses variables from one of the three pairs

$$\{w, x\}, \qquad \{x, y\}, \qquad \{y, z\}.$$

Count $d + 1$ monomials on each pair, then correct for the monomials x^d and y^d which were counted twice. Deduce

$$H_{S/M}(d) = 3(d + 1) - 2 = 3d + 1.$$

Check $d = 0$ separately.

Question 5: Putting the Fibers in One Family. Continue with the weight $\omega = (3, 1, 0, 0)$ and the ideal $M = \langle wy, wz, xz\rangle$.

(a) Construct the weight deformation

$$\mathcal{I}_C = \langle wy - tx^2, \quad wz - t^2xy, \quad xz - ty^2\rangle \subseteq \mathbb{K}[t, w, x, y, z].$$

Check carefully that the powers of t follow from the largest-weight convention.

(b) Write

$$\widetilde{g}_1 = wy - tx^2, \quad \widetilde{g}_2 = wz - t^2xy, \quad \widetilde{g}_3 = xz - ty^2.$$

Verify the three identities

$$\begin{aligned} z\widetilde{g}_1 - y\widetilde{g}_2 &= -tx\widetilde{g}_3, \\ x\widetilde{g}_2 - w\widetilde{g}_3 &= ty\widetilde{g}_1, \\ xz\widetilde{g}_1 - wy\widetilde{g}_3 &= -tx^2\widetilde{g}_3 + ty^2\widetilde{g}_1. \end{aligned}$$

Conclude that the deformed list is a monic Gröbner basis over $\mathbb{K}[t]$ with leading monomial ideal M.

(c) Deduce that the standard monomials of M form a $\mathbb{K}[t]$-basis of $\mathbb{K}[t, w, x, y, z]/\mathcal{I}_C$. This is the promised explicit freeness, and hence flatness, certificate.

(d) At $t = 0$ the fiber is the chain of three lines. If $a \neq 0$, show that the diagonal projective change of coordinates

$$[w : x : y : z] \longmapsto [a^{-3}w : a^{-1}x : y : z]$$

identifies the fiber at $t = a$ with the original twisted cubic. We have therefore produced a flat family whose general fiber is a twisted cubic and whose special fiber is the chain $L_{wx} \cup L_{xy} \cup L_{yz}$.

Question 6: Components and Thickness. Work in a polynomial ring over $\mathbb{K}$.

(a) Show that

$$\langle xy \rangle = \langle x \rangle \cap \langle y \rangle.$$

Both components on the right are prime. What are the corresponding geometric components?

(b) Prove that $\langle x^2 \rangle$ is $\langle x \rangle$-primary. Compare the ideal, its radical, and the visible line it supports. Which nonzero nilpotent in the quotient records the thickness?

(c) Return to $M = \langle wy, wz, xz \rangle$. Why is the decomposition from Exercise 4 already a primary decomposition? Show that M is radical, although it is not prime.

(d) *Another Limit of the Twisted Cubic.* Lecture 3 also encountered the monomial ideal

$$N = \langle x^2, xy, y^2 \rangle = \langle x, y \rangle^2.$$

Show that it is $\langle x, y \rangle$-primary, so its projective zero set is the single line $\mathbb{V}(x, y)$ with a nonreduced structure.

(e) Give w, x, y, z the weights

$$(0, 2, 3, 0)$$

and retain terms of largest weight. Replace the three displayed generators of I_C by their negatives

$$h_1 = x^2 - wy, \qquad h_2 = xy - wz, \qquad h_3 = y^2 - xz.$$

Verify that their weight-leading monomials are x^2, xy, y^2 and that

$$yh_1 - xh_2 = -wh_3, \qquad yh_2 - xh_3 = zh_1.$$

The remaining pair of leading monomials x^2, y^2 is relatively prime. Use Buchberger's criterion for the weight-refined order and the strict comparisons above to conclude

$$\mathrm{in}_\omega(I_C) = N.$$

(f) List the degree-d standard monomials of N as

$$w^a z^b \quad (a + b = d), \qquad xw^a z^b,\ yw^a z^b \quad (a + b = d - 1).$$

Deduce once again that $H_{S/N}(d) = 3d + 1$ and compare with Exercise . Thus this is genuinely another Gröbner degeneration of the twisted cubic, this time to a nonreduced triple structure on a line. Which features agree, and which do not?

(g) The following computation compares these decompositions.

```
restart
needsPackage "PrimaryDecomposition";
S = QQ[w,x,y,z];

M = ideal(w*y,w*z,x*z);
N = ideal(x^2,x*y,y^2);

primaryDecomposition M
primaryDecomposition N
radical N
apply(0..7,d -> hilbertFunction(d,S/N))
```

Predict the radicals and Hilbert-function values before asking the computer. Why does the primary decomposition reveal information which the Hilbert polynomial alone cannot?

Question 7: Bézout from a Hilbert Series. Let $S = \mathbb{K}[x, y, z]$, and let $f, g \in S$ be nonzero homogeneous polynomials of degrees a and b. Assume that f and g have no common irreducible factor. Geometrically, the two projective plane curves have no common component.

(a) Multiplication by f gives an injective map from the degree-$(d-a)$ piece of S into S_d. Show that

$$H_{S/\langle f\rangle}(d) = H_S(d) - H_S(d-a).$$

Equivalently,

$$\mathrm{HS}_{S/\langle f\rangle}(q) = (1-q^a)\,\mathrm{HS}_S(q).$$

(b) Show that multiplication by g is injective on $S/\langle f\rangle$. More explicitly, if $gh \in \langle f\rangle$, use unique factorization and the fact that f and g have no common factor to prove that $h \in \langle f\rangle$.

(c) Apply the same subtraction a second time and use $\mathrm{HS}_S(q) = 1/(1-q)^3$ to obtain

$$\mathrm{HS}_{S/\langle f,g\rangle}(q) = \frac{(1-q^a)(1-q^b)}{(1-q)^3}.$$

(d) Rewrite the numerator as

$$(1-q)^2(1+q+\cdots+q^{a-1})(1+q+\cdots+q^{b-1})$$

and deduce that the Hilbert function of $S/\langle f, g\rangle$ is eventually the constant ab.

(e) Now assume $\mathbb{K}$ is algebraically closed. Since the curves have no common component, their projective intersection is zero-dimensional. Interpret the constant Hilbert polynomial ab as its total length. Explain why this says that curves of degrees a and b meet in ab points when points are counted with multiplicity.

(f) Revisit the circle–parabola and tangent-line examples from Lecture 1. Identify where algebraic closure, projective completion, and nonreduced quotient rings enter this formulation of Bézout's theorem.