PAWS 2026: ALGEBRAIC GEOMETRY VIA COMPUTATION
PROBLEM SET 4

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Question 1: Standard Monomials as Interpolation Data. Let $X = \{p_1, \dots, p_r\} \subseteq \mathbb{K}^n$ be a finite set of distinct $\mathbb{K}$-rational points, set $S = \mathbb{K}[x_1, \dots, x_n]$, and let $I = \mathbb{I}(X)$. Fix a monomial order and a Gröbner basis G of I. Let $B = \{m_1, \dots, m_N\}$ be the standard monomials of $\mathrm{in}_<(I)$.

(a) Prove that the point ideals $\mathfrak{m}_{p_i}$ are pairwise comaximal and apply the Chinese remainder theorem to show that evaluation induces an isomorphism

$$\begin{aligned} \mathrm{ev} : S/I &\longrightarrow \mathbb{K}^r, \\ [f] &\longmapsto \big(f(p_1), \dots, f(p_r)\big). \end{aligned}$$

Notice that algebraic closure of $\mathbb{K}$ is nowhere required here.

(b) Lecture 2 tells us that the classes of the monomials in B form a $\mathbb{K}$-basis of S/I. Deduce that $N = r$. Thus the staircase counts the points, and it does so without being told how many there are.

(c) Write down the evaluation matrix

$$E_B = \big(m_j(p_i)\big)_{1 \le i,j \le r}.$$

Prove that E_B is invertible. Resist the urge to compute a determinant: which two isomorphisms is this matrix comparing?

(d) Given values $b_1, \dots, b_r \in \mathbb{K}$, explain how solving the linear system

$$E_B c = (b_1, \dots, b_r)^{\mathsf{T}}$$

constructs the unique normal form

$$c_1 m_1 + \cdots + c_r m_r$$

having those values on X. An infinite-looking reconstruction problem has become an $r \times r$ linear system; say precisely which step did the reducing.

(e) Changing the monomial order can change the list B. Why can it therefore change the interpolation formula without changing the polynomial function on X? Compare this with the two normal-form bases of the circle–parabola quotient in Lecture 2.

(f) Suppose instead that an ideal J merely satisfies $\mathbb{V}_{\mathbb{K}}(J) = X$ but is not radical. Explain why the evaluation map $S/J \to \mathbb{K}^r$ need not be injective. Which familiar algebraic elements must every point evaluation fail to detect? We have met them before, and we are about to meet them again.

Question 2: Why Lex Eliminates. Prove Theorem 3 from the notes as follows. The whole argument is the one observation about lex order, applied carefully.

(a) Let $0 \neq f \in I \cap \mathbb{K}[x_1, \dots, x_n]$. Since G is a Gröbner basis of I, show that some $g \in G$ satisfies

$$\mathrm{LM}(g) \mid \mathrm{LM}(f).$$

(b) Explain why $\mathrm{LM}(g)$ cannot involve any t_j.

(c) Now suppose some *nonleading* term of g involved a t_j. Use the definition of lex order to show that this term would be larger than the purported leading monomial, which involves only the x-variables—a contradiction. Conclude that $g \in G_x$.

(d) Deduce that every leading monomial of an element of the elimination ideal is divisible by the leading monomial of an element of G_x. Explain why this is exactly the Gröbner-basis condition, and notice that we never had to run a second Buchberger computation.

(e) If G_x turns out to be empty, what does the theorem say about the elimination ideal? Before we make it precise, hazard a geometric guess: what should that mean about the projection?

Question 3: The Twisted Cubic, Now Backwards. Our old companion returns, and this time we approach it from the direction we have been avoiding. In Lecture 1 we began with the affine parametrization $t \mapsto (t, t^2, t^3)$ and computed a kernel by hand, with a rather ad hoc argument about congruences. In Lecture 2 its hidden relation $y^2 - xz$ taught us why Gröbner bases were needed at all. In Lecture 3 the projective parametrization

$$\nu_3 : \mathbb{P}^1 \longrightarrow \mathbb{P}^3, \qquad [s : t] \longmapsto [s^3 : s^2t : st^2 : t^3]$$

showed that the very same quadric was indispensable at infinity. We now recover all of those equations in one stroke, by elimination.

(a) For the affine parametrization, form the graph ideal

$$L_{\text{aff}} = \langle x - t, y - t^2, z - t^3 \rangle \subseteq \mathbb{K}[t, x, y, z].$$

Identify $L_{\text{aff}} \cap \mathbb{K}[x, y, z]$ with the kernel from Lecture 1. Compare the length of this argument with the one we gave there.

(b) Without computing a Gröbner basis, verify that

$$x^2 - y, \qquad xy - z, \qquad y^2 - xz$$

belong to this elimination ideal. The third is a consequence of the first two; why does that not make it useless? (You have now been asked this three times, in three different disguises.)

(c) Now regard the four homogeneous coordinates of ν_3 as a polynomial map $\mathbb{K}^2 \to \mathbb{K}^4$ and form

$$L_{\text{proj}} = \langle w - s^3, x - s^2t, y - st^2, z - t^3 \rangle \subseteq \mathbb{K}[s, t, w, x, y, z].$$

Verify directly that the three quadrics

$$x^2 - wy, \qquad xy - wz, \qquad y^2 - xz$$

lie in $L_{\text{proj}} \cap \mathbb{K}[w, x, y, z]$.

(d) Arrange these quadrics, up to sign, as the 2×2 minors of

$$\begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix}.$$

Which entry was 1 in the affine matrix from Lecture 2, and why did it become w? This is homogenization happening in front of us, in a form which is hard to misread.

(e) Predict the eliminated ideal before running the following block. Then compare the output with the ideal found in Lecture 3.

```
restart
needsPackage "Elimination";
R = QQ[s,t,w,x,y,z,MonomialOrder=>Lex];

L = ideal(w-s^3,x-s^2*t,y-s*t^2,z-t^3);
G = gb L;
K = eliminate({s,t},L);

trim K
J = ideal(x^2-w*y,x*y-w*z,y^2-x*z);
K == J
```

Why are s and t placed first in the ring, and what would go wrong if they were placed last? What does the output `true` certify that a visual comparison of two generating lists would not?

(f) The calculation took place in the affine cone $\mathbb{K}^4$, whereas ν_3 maps into projective space. Conclude that homogeneous generators of the kernel define genuine projective equations. Assuming $\mathbb{K}$ is infinite, use the work in Lecture 3 to identify J with the full homogeneous vanishing ideal of $\nu_3(\mathbb{P}^1)$.

(g) Finally, compare the two routes. In Lecture 3 we computed a degree-compatible affine Gröbner basis and then homogenized. Here we homogenized the parametrization and then eliminated its parameters. Why should two such different-looking procedures arrive at the same ideal? Which route starts from equations, and which starts from points manufactured by a map?

Question 4: Drawing the Circle with a Line. Assume that $\mathrm{char}(\mathbb{K}) \neq 2$ and consider

$$C = \mathbb{V}(x^2 + y^2 - 1) \subseteq \mathbb{K}^2.$$

The point $P = (-1, 0)$ lies on C. A line of slope t through P has equation

$$y = t(x + 1).$$

Draw the picture before doing any algebra; the entire exercise is visible in it.

(a) Substitute the line equation into the circle equation and verify the factorization

$$x^2 + t^2(x + 1)^2 - 1 = (x + 1)\big((1 + t^2)x + t^2 - 1\big).$$

Which factor records the intersection point we already knew about? This is the crucial move: we knew one root in advance, so we may divide it out.

(b) When $1 + t^2 \neq 0$, solve for the second intersection point and obtain

$$\phi(t) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2}\right).$$

Verify directly that $\phi(t) \in C$.

(c) Let $(x, y) \in C$ with $x \neq -1$. The line joining P to (x, y) has slope

$$t = \frac{y}{x + 1}.$$

Use $x^2 + y^2 = 1$ to show that

$$1 + t^2 = \frac{2}{x + 1}.$$

Deduce that this denominator is nonzero and that substituting this t into ϕ recovers (x, y). Thus the geometric construction has an algebraic inverse, and both directions are given by ratios of polynomials.

(d) Conclude that ϕ gives a bijection

$$\{t \in \mathbb{K} : 1 + t^2 \neq 0\} \longrightarrow C(\mathbb{K}) \setminus \{(-1, 0)\}.$$

Over $\mathbb{R}$, which values of t must be excluded? Over $\mathbb{C}$, which additional values appear, and what geometric feature will projective space eventually assign to them?

(e) The point P was reached by no finite slope at all. Explain geometrically why it ought to correspond to the vertical line through P, or informally to $t = \infty$.

(f) Clear denominators in the identity

$$(x, y, 1) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2}, 1\right)$$

and compare the resulting quadratics with the classical Pythagorean-triple formula. You have very likely met these three expressions before, in a course which had nothing to do with algebraic geometry.

Question 5: The Missing Parameter Goes to Infinity. Assume that $\mathrm{char}(\mathbb{K}) \neq 2$. Let

$$\overline{C} = \mathbb{V}_{\mathbb{P}^2}(X^2 + Y^2 - Z^2)$$

be the projective circle. Define

$$\Phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^2, \qquad [s : t] \longmapsto [s^2 - t^2 : 2st : s^2 + t^2].$$

(a) Verify that all three coordinates have no common zero in $\mathbb{P}^1$. Where do you use $\mathrm{char}(\mathbb{K}) \neq 2$? Conclude that Φ is defined everywhere.

(b) Substitute the coordinates into $X^2 + Y^2 - Z^2$ and show that $\Phi(\mathbb{P}^1) \subseteq \overline{C}$.

(c) On the chart $s \neq 0$, write $u = t/s$. Show that the part of the image with $Z \neq 0$ recovers the affine formula from Exercise 4, so that nothing has been lost in the passage to $\mathbb{P}^1$.

(d) Evaluate $\Phi([0 : 1])$. Which missing affine point does this recover, and does the answer match the geometric prediction you made in Exercise 4(e)?

(e) If $s^2 + t^2 = 0$, what kind of projective point does $\Phi([s : t])$ produce? Explain what became of the forbidden affine parameters $t = \pm\sqrt{-1}$ when $\mathbb{K}$ happens to contain them. The denominators did not vanish into nothing; they went somewhere.

(f) On the open subset $U_+ \subseteq \overline{C}$ where $Z + X \neq 0$, verify that

$$[X : Y : Z] \longmapsto [Z + X : Y]$$

is inverse to Φ wherever $s \neq 0$.

(g) On the open subset $U_- \subseteq \overline{C}$ where $Z - X \neq 0$, verify that

$$[X : Y : Z] \longmapsto [Y : Z - X]$$

is inverse to Φ wherever $t \neq 0$. Show that U_+ and U_- cover $\overline{C}$. On their overlap, use

$$(Z + X)(Z - X) = Y^2$$

to prove that $[Z + X : Y] = [Y : Z - X]$. The two partial inverses therefore glue into a single map $\overline{C} \to \mathbb{P}^1$ defined at every point, with no exceptions to apologize for.

(h) Conclude that Φ is not merely a birational parametrization: it is an isomorphism

$$\mathbb{P}^1 \cong \overline{C}.$$

Compare this global statement with the affine parametrization, where one point and a couple of denominator values had to be excluded by hand.