PAWS 2026: ALGEBRAIC GEOMETRY VIA COMPUTATION
PROBLEM SET 1

TOREN D'NELLY-WARADY, AIDAN HENNESSEY, SMITA RAJAN, JAMES RAWSON

1. Theoretical Problems

Question 1: A Circle & A Parabola. Consider the following system of two polynomial equations:

$$f_1 = x^2 - y = 0, \qquad f_2 = x^2 + y^2 - 1 = 0.$$

Since f_1 defines a parabola and f_2 the unit circle at the origin we are looking at the intersection of two very familiar shapes.

(1) Substitute the first equation into the second and show that every solution satisfies
$$x^4 + x^2 - 1 = 0.$$
The roots of this new polynomial are the possible x-coordinates of the intersection. What information was eliminated in passing from two equations in x, y to one equation in x?

(2) Consider $x^4 + x^2 - 1$ over $\mathbb{R}$ and $\mathbb{C}$. How many solutions are there over each field? Sketch the picture.

(3) Assume $\mathbb{K}$ is infinite and consider the hyperbola
$$H = \left\{(x, y) \in \mathbb{K}^2 \mid xy = 1\right\}.$$
Show that the projection of H to the x-axis is $\mathbb{K}^\times$. Prove that the only polynomial in $\mathbb{K}[x]$ which vanishes on this image is the zero polynomial.

Question 2: From Equations to Ideals. Let $\mathcal{F} \subseteq \mathbb{K}[x_1, \ldots, x_n]$ and let $I, J \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be ideals.

(1) Prove that $\mathbb{V}(\mathcal{F}) = \mathbb{V}(\langle \mathcal{F} \rangle)$.

(2) Prove that $\mathbb{V}$ reverses inclusions: if $I \subseteq J$, then $\mathbb{V}(J) \subseteq \mathbb{V}(I)$.

(3) Describe the following varieties and, when possible, draw the corresponding picture. Over $\mathbb{C}$, a literal sketch is likely difficult, so you may need to be creative.
$$\mathbb{V}_\mathbb{R}(x^2 + y^2 - 1), \quad \mathbb{V}_\mathbb{R}(x^2 - y^2), \quad \mathbb{V}_\mathbb{R}(x^2 + y^2),$$
$$\mathbb{V}_\mathbb{C}(x^2 + y^2 - 1), \quad \mathbb{V}_\mathbb{C}(x^2 - y^2), \quad \mathbb{V}_\mathbb{C}(x^2 + y^2).$$
An *affine plane conic* is the zero set of a polynomial of degree 2 in two variables, and thus, the six varieties above are all examples of plane conics.

(4) Which of the defining polynomials in the previous part are irreducible over $\mathbb{R}$, but reducible over $\mathbb{C}$? For these examples, how do the varieties over $\mathbb{R}$ and $\mathbb{C}$ differ?

(5) Prove that for any polynomials $f, g \in \mathbb{K}[x_1, \dots, x_n]$ the following identity holds:
$$\mathbb{V}(fg) = \mathbb{V}(f) \cup \mathbb{V}(g).$$
Thus factorization is an algebraic signal that the corresponding variety can be broken into (simpler) pieces.

Question 3: Radical Ideals. Let R be a ring and $I \subseteq R$ an ideal.

(1) Show that $I \subseteq \sqrt{I}$ and that $\sqrt{\sqrt{I}} = \sqrt{I}$.

(2) Show that $\sqrt{I}$ is an ideal.

(3) Let $I \subseteq \mathbb{K}[x_1, \dots, x_n]$ be an ideal. Prove that $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$.

(4) Compute $\sqrt{\langle x^3y^2 \rangle} \subseteq \mathbb{K}[x, y]$. Compare the zero sets of $\langle x^3y^2 \rangle$ and its radical.

(5) Give three different ideals in $\mathbb{K}[x, y]$ which define the x-axis. Which, if any, of your ideals are radical?

Question 4: From a Parametrization to Equations: The Affine Twisted Cubic. Consider the polynomial map
$$F : \mathbb{K} \longrightarrow \mathbb{K}^3 \qquad \text{given by} \qquad t \longmapsto (t, t^2, t^3),$$
and let C denote the image of F. This curve is called the *affine twisted cubic*; the reason for the name will become clearer when we introduce projective space. Substitution gives the ring homomorphism
$$\begin{array}{ccc} \mathbb{K}[x, y, z] & \xrightarrow{\quad F^* \quad} & \mathbb{K}[t] \\ x & \longmapsto & t \\ y & \longmapsto & t^2 \\ z & \longmapsto & t^3 \end{array}$$
which is the identity on $\mathbb{K}$.

(1) Verify that $y - x^2$ and $z - xy$ belong to $\ker(F^*)$. Let $J = \langle y - x^2, z - xy \rangle$. Explain why this proves that $J \subseteq \ker(F^*)$ and $C \subseteq \mathbb{V}(J)$.

(2) Conversely, let $(a, b, c) \in \mathbb{V}(J)$. Show that $b = a^2$ and $c = a^3$, and hence that $(a, b, c) = F(a)$.

(3) Conclude that $C = \mathbb{V}(J)$. Hence, in this example, the image of the parametrization is already algebraic. This equality holds over every field.

(4) The previous part showed $\mathbb{V}_{\mathbb{K}}(J) = \mathbb{V}_{\mathbb{K}}(\ker(F^*))$. Give a reason why we cannot conclude $\sqrt{J} = \sqrt{\ker(F^*)}$ from this.

(5) Show that $\mathbb{I}(C) = \ker(F^*) = J$. Either do your own thing, or follow the steps outlined in the lecture notes.

Question 5: Coordinate Rings, Nilpotents, and Points.

Let $S = \mathbb{K}[x_1, \ldots, x_n]$.

(1) Let $I \subseteq S$. Show that the nilpotent elements of S/I are precisely the classes $f + I$ with $f \in \sqrt{I}$. Deduce that S/I is reduced if and only if I is radical.

(2) Let X be a nonempty algebraic variety. A nonempty algebraic variety $X \subset \mathbb{K}^n$ is *reducible* if and only if there exist algebraic varieties $X_1, X_2 \subset \mathbb{K}^n$, with $X_i \neq X$ for $i = 1, 2$, such that $X = X_1 \cup X_2$. A variety is *irreducible* if it is not reducible. Using $\mathbb{V}(fg) = \mathbb{V}(f) \cup \mathbb{V}(g)$, prove that

$$\begin{aligned} X \text{ is irreducible} \quad &\Longleftrightarrow \quad \mathbb{I}(X) \text{ is prime} \\ &\Longleftrightarrow \quad \mathbb{K}[X] \text{ is an integral domain.} \end{aligned}$$

Notice that this equivalence does not require $\mathbb{K}$ to be algebraically closed.

Hint: For the reverse implication, if $X = X_1 \cup X_2$ with both $X_i \subsetneq X$, choose $f_i \in \mathbb{I}(X_i) \setminus \mathbb{I}(X)$.

(3) Let $I \subseteq S$ and suppose that $X = \mathbb{V}(I)$ is nonempty. Show that the natural map $q_I : S/I \longrightarrow \mathbb{K}[X]$ has kernel $\mathbb{I}(X)/I$.

(4) Show that every $p \in X$ induces an evaluation homomorphism $\mathrm{ev}_p : S/I \longrightarrow \mathbb{K}$ and that

$$\ker(q_I) = \bigcap_{p \in X} \ker(\mathrm{ev}_p).$$

(5) Assuming that $\mathbb{K}$ is algebraically closed, use the Nullstellensatz to conclude that this intersection is precisely the set of nilpotent elements of S/I.

Question 6: A Double Point from Tangency. Let $A_1 = \mathbb{K}[x]/\langle x \rangle$ and $A_2 = \mathbb{K}[x]/\langle x^2 \rangle$. Notice that both $\langle x \rangle$ and $\langle x^2 \rangle$ define the single point $\{0\} \subseteq \mathbb{K}$.

(1) Prove that $\overline{x} \neq 0$ in A_2 while $\overline{x}^2 = 0$. Identify the kernel of the natural map $A_2 \longrightarrow A_1 \cong \mathbb{K}[\{0\}]$, and explain why evaluation at the unique point cannot detect $\overline{x}$.

(2) The preceding algebra occurs geometrically when curves meet tangentially. The parabola $y = x^2$ is tangent to the line $y = 0$ at the origin. Set

$$I_{\tan} = \langle x^2 - y, y \rangle \subseteq \mathbb{C}[x, y].$$

Show that $\mathbb{C}[x, y]/I_{\tan} \cong \mathbb{C}[x]/\langle x^2 \rangle$.

(3) Explain why the underlying intersection consists of only the origin, while the quotient ring has dimension 2 as a $\mathbb{C}$-vector space. Since this is the only intersection point, this dimension is its intersection multiplicity: the origin is therefore *counted twice.*

2. Computational Problems

Question 7: First *Macaulay2* Session. Let us return to the circle–parabola equations, which are defined by the ideal $I = \langle x^2 - y, x^2 + y^2 - 1 \rangle \subset \mathbb{K}[x, y]$. Enter the following commands one line at a time and compare every output with Exercise 1.

```
restart
R = QQ[x,y];
f1 = x^2-y;
f2 = x^2+y^2-1;
I = ideal(f1,f2); -- encode the intersection as an ideal

R
gens I
f2-f1 -- eliminate x
```

The command `R = QQ[x,y]` creates the polynomial ring $R = \mathbb{Q}[x, y]$. After running this line, we can use the variables x and y to construct polynomials in R. For example `f1 = x^2-y` constructs the polynomial $f_1 = x^2 - y$ and *Macaulay2* knows this is a polynomial in R. The command `I = ideal(f1,f2)` creates the ideal $I = \langle f_1, f_2 \rangle \subset R$. In order to see the generators of I we use the command `gens I`, which displays the chosen generators as a row matrix.

(1) After running the above code explain the three displayed outputs.

(2) Write code that determines whether y^2+y-1 lies in the ideal I. What does this mean geometrically? *Hint:* One direct test is `isSubset(ideal(y^2+y-1),I)`, but can you find a more direct method?

Question 8: A Second *Macaulay2* Session. Recall the parametrization of the affine twisted cubic from Question 4, $F : \mathbb{K} \to \mathbb{K}^3, t \mapsto (t, t^2, t^3)$, and the corresponding map $F^* : \mathbb{K}[x, y, z] \to \mathbb{K}[t]$.

(1) In *Macaulay2*, start a new session with the `restart` command. Create the rings $A := \mathbb{K}[x, y, z]$ and $T := \mathbb{K}[t]$, and use the `map` command to construct the ring homomorphism F^*, and assign it to the variable `phi`.

(2) What do you think the output of `phi(y - x^2)` will be? What about `phi(z - x*y)`? Run these commands and check they agree with what you expect.

(3) Use the `ker` command to compute the kernel `K` of `phi`. Simplify the generators with `trim`. Does the output match those computed in Question 4? If not, are they equivalent? Can you check this in *Macaulay2*?

(4) Compute the values of `phi` applied to `y - x^2, z-x*y, x*z - y^2, x`. To avoid repeating commands, try constructing a list and using the `apply` command. This iterates a function over a list and creates a new list with the output of the function.

(5) Now print pairs `(g, phi(g))` as `g` varies over that list. This can be achieved with the `scan` command, which is similar to `apply` except it does not store the outputs. The command `scan` is more useful when computing with functions that have an output effect.