Computational Algebraic Geometry

Lecture V - What an Initial Ideal Remembers

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Algebra	⟷	Geometry
"equations"		"solutions"
"ideals"		"varieties"
"homog. ideals"		"proj. varieties"

- If we fix a term order $\leq$ on $\mathbb{K}[x_1, \ldots, x_n] = S$, $I \subseteq S$ an ideal the initial ideal of I w/r/t $\leq$

$$in_{\leq}(I) = \langle in_{\leq}(f) \mid f \in I \rangle$$

this led us to gröbner basis and solved:

1. Detect when $V(I) = \emptyset$ / ideal membership problem
2. compute proj. closures
3. Compute $\ker(F^*)$ using elimination

⋮

Main Question:

1. What does $in_{\leq}(I)$ have to do with I algebraically?
2. What does $V(in_{\leq}(I))$ have to do with $V(I)$?

$S = \mathbb{k}[x_0, \ldots, x_n]$ $\quad (\sim \mathbb{P}^n)$

The standard grading on S, is a decomposition of S as an abelian group

$$S \cong \bigoplus_{d \geq 0} S_d$$

where

S_d = $\mathbb{k}$-vector space spanned by monomials of degree exactly d.

(i.e. every $f \in S$ can we written uniquely as a finite sum of elements in S_d for varying d)

with property that

$$S_d \cdot S_e \subseteq S_{d+e}$$

we call elements of S_d homog. of degree d

$x^2 + y^2 \in S_2$ $\qquad$ $x^2 + y^3 \notin S_2$ or S_3

$x^2 + 1 \in S_d$ for any d.

- **Def:** An ideal $I \subseteq S$ is homog. iff every element $f \in I$ can we be written as a sum of it homog. pieces contained in I

$$I \cong \bigoplus_{d \geq 0} (I \cap S_d)$$

• **Lemma:** If $I \subseteq S$ is homogenous if

$$(S/I)_d = \{ \bar{f} \in S/I \mid f \in S_d \}$$

then as $\mathbb{K}$-vector spaces

$$(S/I)_d \cong S_d/I_d$$

where $I_d = I \cap S_d$

Cor: $S/I \cong \bigoplus_{d \geq 0} (S/I)_d$ as a graded ring.

Ex: $\mathbb{K}[x]/\langle x-1 \rangle$ then $\bar{x} = \bar{1}$ does $\bar{x}$ have degree 1 or deg 0? Can't happen when I is homog.

• **Def:** Let $I \subseteq S$ be a homogenous ideal

1) The Hilbert Function of S/I is the function

$$\mathbb{Z}_{\geq 0} \xrightarrow{H_{S/I}} \mathbb{Z}$$

$$d \longmapsto \dim_{\mathbb{K}} (S/I)_d$$

2) The Hilbert series of S/I is the formal generating function

$$HS_{S/I}(t) = \sum_{d \geq 0} H_{S/I}(t)\, t^d$$

Note $\dim_{\mathbb{K}} (S/I)_d = \dim_{\mathbb{K}} (S_d/I_d)$

$= \dim_{\mathbb{K}} S_d - \dim_{\mathbb{K}} I_d$

Lemma: 1) $H_S(d) = \binom{n+d}{d}$ = # monomials of degree d in $n+1$ variables

2) If $f \in S$ is homog. of degree e and $f \neq 0$

$$H_{S/\langle f \rangle}(d) = \binom{n+d}{d} - \binom{n+d-e}{n}$$

Ex: If $C \subseteq \mathbb{P}^2$ is a plane conic ie $C = V(Q)$

For $Q \in \mathbb{K}[x,y,z]$ homog. nonzero deg 2 then

$$H_C(d) = 2d + 1$$

- **Thm:** If $I \subseteq S$ is homog. and $<$ is a monomial order on S then

$$H_{S/I}(d) = H_{S/in_<(I)}(d) \qquad \forall d \geq 0$$

- **Thm:** If $I \subseteq S$ is a homogenous ideal then there exists a polynomial $P_{S/I}(d) \in \mathbb{Q}[d]$ such that

$$H_{S/I}(d) = P_{S/I}(d) \qquad \forall d \gg 0$$

Moreover, if $V(I) \subseteq \mathbb{P}^n$ has dimension r then

$$P_{S/I}(d) = \frac{\deg(I)}{r!} d^r + \text{lower order terms}$$

The polynomial $P_{S/I}$ is called the Hilbert polynomial

- **Cor.** (Reiman-Roch): If $I \subseteq S$ defines a curve then

$$P_{S/I}(d) = \deg(I)\, d + 1 - P_a$$

where P_a is the arithmetic genus, and if the curve is smooth $P_a = g$ the geometric genus.

- **Def**: Fix $\bar{w} \in (w_0, \dots w_n) \in \mathbb{Z}^{n+1}$

$$wt_{\bar{w}}(x^{\bar{a}}) = \bar{w} \cdot \bar{a} = w_0 a_0 + \dots + w_n a_n$$

For a polynomial $f = \sum c_{\bar{a}} x^{\bar{a}}$

$$W_{\bar{w}}(f) = \max\{ \bar{w} \cdot \bar{a} \mid c_{\bar{a}} \neq 0 \}$$

$$in_{\bar{w}}(f) = \sum_{\bar{a} \cdot \bar{w} = W_{\bar{w}}(f)} c_{\bar{a}} x^{\bar{a}}$$

* This is not necessarily a monomial *

$$in_{\bar{w}}(I) = \langle in_{\bar{w}}(f) \mid f \in I \rangle$$

Given a polynomial $f = \sum c_\alpha x^{\bar{\alpha}}$ define

$$\tilde{f}(t, x_0, \dots x_n) = \sum_\alpha c_\alpha t^{w_{\bar{w}}(f) - \bar{w}\bar{\alpha}} x^{\bar{\alpha}}$$

$$= t^{w_{\bar{w}}(f)} f(t^{-w_0} x_0, \dots \; t^{-w_n} x_n)$$

Ex: ① $f = xy - z^2$ $\quad \bar{w} = (1,0,0)$

$\tilde{f} = xy - tz^2$

$in_{\bar{w}}(f) = xy$

$t = 0 \rightsquigarrow xy = in_{\bar{w}}(f)$

$t = 1 \rightsquigarrow xy - z^2 = f$

② $g = x^2 - yz$ $\quad \bar{w} = (1,1,0)$

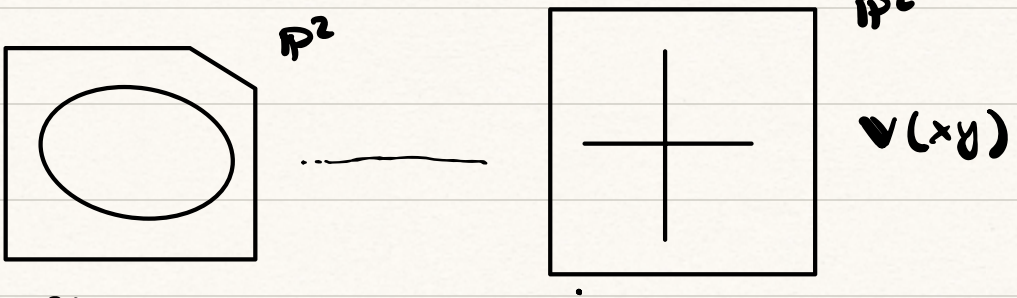
$\tilde{g} = x^2 - tyz$

$in_{\bar{w}}(g) = x^2$

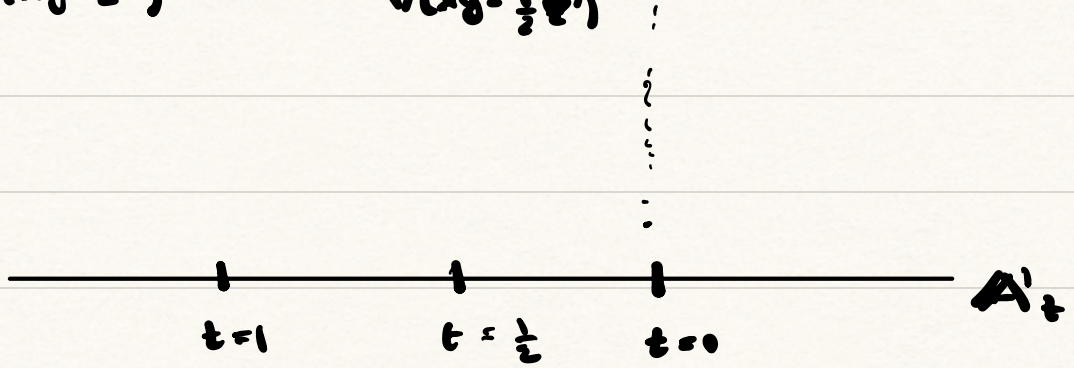
$t = 0 \rightsquigarrow x^2 = in_{\bar{w}}(f)$

$t = 1 \rightsquigarrow x^2 - yz = g.$

By introducing the variable t we built a "family" of varieties over $\mathbb{A}^1_t$ with the fibers being varieties in the variables $x_0, \dots x_n$ i.e. subvarieties of $\mathbb{P}^n$



$V(xy - z^2)$



- **Thm:** Let $I \subseteq S$ be homog. and $\bar{w} \in \mathbb{Z}^{n+1}$ If $G = (g_1, \dots g_s)$ is a Grobner basis for $in_{\leq w}(I)$ then

$$\mathcal{I} = \langle \tilde{g}_1, \dots \tilde{g}_s \rangle \subseteq \mathbb{K}[t][x_0, \dots x_n]$$

① The fibre at $t=0$ is $\mathbb{V}(in_{\leq}(I))$

② at every every fibre $t = a \neq 0$ $\mathbb{V}(I)$ is sent to ithe fibre by the change of coordinates $x_i \longmapsto a^{-w_i} x_i$

③ the standard monomials of $in_w(I)$ form a $\mathbb{K}[t]$-basis of $\mathbb{K}[t][x_0, \dots x_n]/\mathcal{I}$

In particular, this is a "flat" family and the Hilbert function / polynomial is the same for all fibres.