Computational Algebraic Geometry

Lecture IV - Interpolation, Implicitization, Rationality.

Juliette Bruce (Dartmouth)

- So far we have focused



- A big tool on the equation side has been Gröbner bases

① Given finitely many points can we find equations that satisfy them (defining ideal) [Interpolation]

② Given a parameterization $t \longmapsto (g_1(t), \dots g_n(t))$ Can we find the defining ideal. [implicitization]

③ When do parameterizations exist? [Rationality]

- **Ex**: Given points $a_1, \dots a_r \in \mathbb{K}$ (distinct?) can we find polys in $\mathbb{K}[x]$ defining $a_1, \dots, a_r$.

Yes: $\prod_{i=1}^{r} (x - a_i) = f(x)$ vanishes exactly at a_i!

$$I(\{a_1, \dots a_r\}) = \langle 0 \rangle \overset{CRT}{=} \bigcap_{i=1}^{r} \langle x - a_i \rangle.$$

↳ defines a_i.

There is a map

$$\frac{k[x]}{\langle 0 \rangle} \xrightarrow[\sim]{ev} k^r$$

$$[f] \longmapsto (f(a_1), \dots, f(a_r))$$

$1, x, x^2, \dots x^{r-1}$ basis

$l_1, \dots l_r$

$$l_i(x) = \prod_{j \neq i} \frac{x - a_j}{a_i - a_j}$$

$$l_i(a_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & i = j \end{cases} = \delta_{ij}$$

General Answer ①: Suppose $X = \{P_1, \dots P_r\} \subseteq k^n$ (distinct)

① Then $I(X) = \bigcap_{i=1}^{r} \langle \bar{x} - P_i \rangle$

$\langle x_1 - P_{i1}, x_2 - P_{i2}, \dots x_n - P_{in} \rangle$

② If G is any Grobner basis for $I(X)$ and B is the std monomials for G.

$$\frac{k[x_1, \dots x_n]}{I(X)} \xrightarrow[\sim]{ev_B} k^r$$

$$ev_B = (m_j(P_i))$$

where $\{m_j\} = B$.

II. Implicitization: Suppose $F: \Bbbk^m \longrightarrow \Bbbk^n$

given by $\bar{a} \longmapsto (g_1(\bar{a}), \dots g_n(\bar{a}))$

where $g_1, \dots g_n \in \Bbbk[t_1, \dots t_m]$

$\Rightarrow F^*: \Bbbk[x_1, \dots x_n] \longrightarrow \Bbbk[t_1, \dots t_m]$

$x_i \longmapsto g_i$

- **Question**: Can we find $\ker(F^*)$

$R = \Bbbk[t_1, \dots t_m, x_1, \dots x_n]$

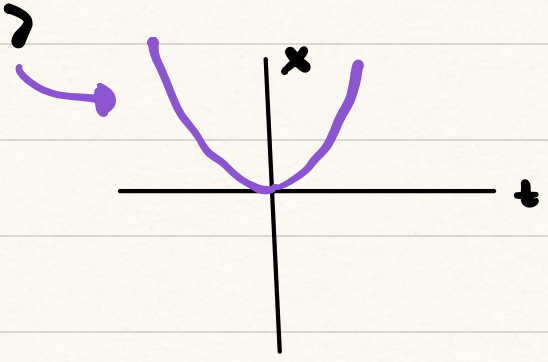
$J_F = \langle x_1 - g_1, x_2 - g_2, \dots, x_n - g_n \rangle$ "Graph Ideal of F"

$V(J_F) \subseteq \Bbbk^{m+n} \cong \Bbbk^m \times \Bbbk^n$

$= \{(\bar{a}, F(\bar{a})) \mid \bar{a} \in \Bbbk^m\}$ graph of F!

$t \xmapsto{F} t^2$ $\qquad \mathbb{R} \longrightarrow \mathbb{R}$

$J_F = \langle x - t^2 \rangle$



Thm: $\ker(F^*) = J_F \cap \mathbb{k}[x_1, \dots x_n]$

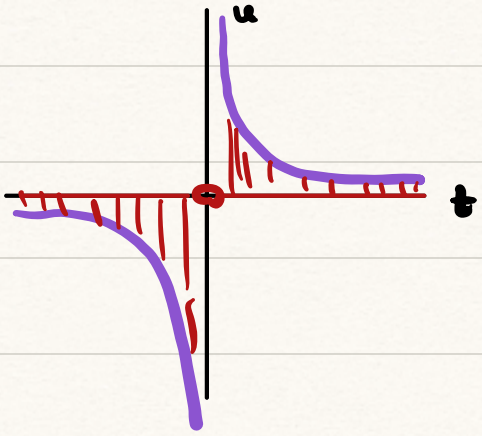
↑ this notation seems bc J_F and $\mathbb{k}[x_1, \dots x_n]$ are subset of $\mathbb{k}[t_1, \dots t_m, x_1, \dots x_n]$

R

Thm: Consider $\mathbb{k}[t_1, \dots t_m, x_1, \dots x_n]$ w/ LEX where $t_1 > t_2 > \dots > t_m > x_1 > x_2 > \dots x_n$. If $I \subseteq R$ is an ideal and G is a Gröbner basis for I then $G \cap \mathbb{k}[x_1, \dots x_n]$ is Gröbner basis $I \cap \mathbb{k}[x_1, \dots x_n]$.

↑ Elimination Theory.

- **Ex:** $\mathbb{V}(tu - 1) \subseteq \mathbb{k}^2$



$\pi : \mathbb{V}(tu-1) \longrightarrow \mathbb{k}$

$u = 1/t \quad (u,t) \longmapsto$

$\text{img}(\pi) = \mathbb{k} \setminus \{0\}$

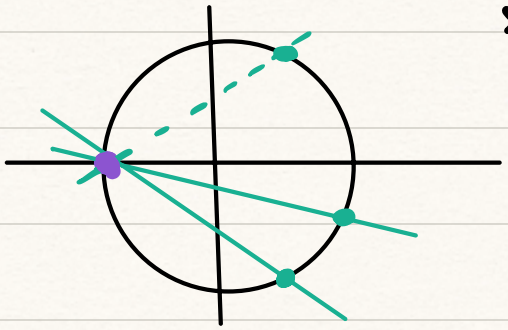
But: $\langle tu - 1 \rangle \cap \mathbb{k}[t] = \langle 0 \rangle$

↑ vanish on $\text{img}(\pi)$

- **Thm:** If $\mathbb{k}$ is infinite $\overline{F(\mathbb{k}^m)} = \mathbb{V}(\ker(F^*))$

- **Cor:** $\mathbb{V}(y^2 - x^3 + x)$ has no parameterization.

$x^2+y^2=1$ $P=(-1,0)$



$$\mathbb{R} \longrightarrow \mathbb{R}^2$$

$$t \longmapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

plug in $y=t(x+1)$

Gives a bijection

- If X and Y are algebraic varieties a rational map $X \longrightarrow Y$ is a function given on a nonempty open subset by rational equation.

- Two varieties X and Y are birational $\Leftrightarrow$ rational maps both ways that mutal inveses on open dense sets.

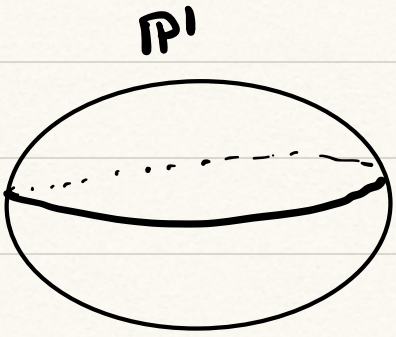
- A variety X is rational iff its birational $\mathbb{K}^n$ or $\mathbb{P}^n$

  $\Leftrightarrow$ $x^2+y^2=1$ is rational $\mathbb{R}/\mathbb{C}/\mathbb{Q}$

- $y^2=x^3-x$ is not rational. $\rightsquigarrow$ Elliptic Curve



$g=1$



$\mathbb{P}^1$

$g=0$ 14