Computational Algebraic Geometry

Lecture I - Equations & Geometry

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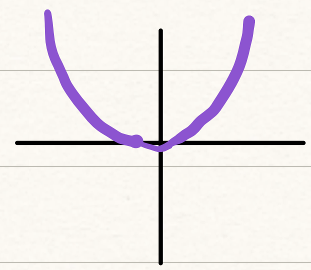
Computational algebraic geometry is an approach to the study of systems of polynomial equations that's seriously and is informed by <u>computations</u>

↳ By hand and By Computer (Macaulay2)

- <u>**Ex**</u>: <u>Equation</u>: $y = x^2$ <u>Graph</u>:



$\{(x,y) \in \mathbb{R}^2 \mid x^2 = y\}$

$$\begin{tikzcd} \text{"Polynomial Equations"} \arrow[r, bend left, "V"] & \text{"Solution Sets"} \arrow[l, bend left, "I"] \end{tikzcd}$$

(Algebra) (Geometric)

$\mathbb{V}$-direction: Is asking us to "solve" a system of polynomial equations. For example:

$$\begin{cases} f_1 = x^2 - y = 0 \\ f_2 = x^2 + y^2 - 1 = 0 \end{cases}$$

1. Does any solution exist?
2. Can we find all the equations that hold on every solution?
3. Is the set of solutions finite? Pos. Dim? Multiplicity?
4. Do the solutions break into nice pieces?
5. Which solutions can be parameterized

• **Def:** If $\mathcal{F} \subseteq \mathbb{k}[x_1, \dots, x_n]$. The vanishing set of $\mathcal{F}$ is

$$\mathbb{V}_{\mathbb{k}}(\mathcal{F}) := \{ p \in \mathbb{k}^n \mid f(p) = 0 \text{ for all } f \in \mathcal{F} \}$$

Note: • The set $\mathcal{F}$ can be arbitrary.

• It depends on the field $\mathbb{k}$ $\quad \mathbb{V}_{\mathbb{R}}(x^2+1)$ vs $\mathbb{V}_{\mathbb{C}}(x^2+1)$

• When the field is clear we write $\mathbb{V}(\mathcal{F})$

• **Def:** A subset of $\mathbb{k}^n$ of the form $\mathbb{V}(\mathcal{F})$ for some subset $\mathcal{F} \subseteq \mathbb{k}[x_1, \dots, x_n]$ is called an algebraic set / algebraic variety.

Lemma: 1) $\mathbb{V}(\mathcal{F}) = \mathbb{V}(\langle \mathcal{F} \rangle)$

2) If $\mathcal{F} \subseteq \mathcal{S}$ then $\mathbb{V}(\mathcal{F}) \supseteq \mathbb{V}(\mathcal{S})$

3) $\mathbb{V}(fg) = \mathbb{V}(f) \cup \mathbb{V}(g)$

By (1) above $\mathbb{V}$ gives a map of sets

$$\left\{ \begin{matrix} \text{ideals in} \\ \mathbb{k}[x_1, \dots, x_n] \end{matrix} \right\} \xrightarrow{\ \mathbb{V}\ } \left\{ \begin{matrix} \text{algebraic} \\ \text{varieties} \\ \text{in } \mathbb{k}^n \end{matrix} \right\}$$

$$I \longmapsto \mathbb{V}(I)$$

Ex: $\mathbb{V}_{\mathbb{C}}(x) = \{\bar{0}\}$

$\mathbb{V}_{\mathbb{C}}(x^2) = \{\bar{0}\}$

More generally if $p \in \mathbb{k}^n$ then $(f(p))^k = 0 \iff f(p) = 0$ since $f(p) \in \mathbb{k}$ and $\mathbb{k}$ is a field.

Def: If $I \subseteq \mathbb{k}[x_1, \dots, x_n]$ is an ideal the radical of I is

$$\sqrt{I} = \{ f \in \mathbb{k}[x_1, \dots, x_n] \mid f^k \in I \text{ for some } k \geq 1 \}$$

We say I is radical iff $I = \sqrt{I}$.

• **Lemma**: 1) $I \subseteq \sqrt{I}$

2) If I is an ideal then $\sqrt{I}$ is an ideal.

3) $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$

• **Thm**: (Hilbert's Nullstellensatz): If $\mathbb{k}$ is algebraically closed then the map of sets $I \longmapsto \mathbb{V}(I)$ gives a bijection

$$\left\{ \begin{matrix} \text{radical ideals} \\ \text{in } \mathbb{k}[x_1, \dots x_n] \end{matrix} \right\} \xrightarrow{\ \mathbb{V}\ } \left\{ \begin{matrix} \text{algo varieties} \\ \text{in } \mathbb{k}^n \end{matrix} \right\}$$

• **Def**: For any subset $X \subseteq \mathbb{k}^n$ its vanishing ideal

$$\mathbb{I}(X) = \{ f \in \mathbb{k}[x_1, \dots x_n] \mid f(p) = 0 \text{ for all } p \in X \}$$

Lemma: 1) $\mathbb{I}(X)$ is an ideal

2) $\mathbb{I}(X)$ is a radical ideal

3) $X \subseteq Y$ then $\mathbb{I}(X) \supseteq \mathbb{I}(Y)$

4) $X \subseteq \mathbb{V}(\mathbb{I}(X))$

5) $\mathbb{V}(\mathbb{I}(X)) = X$ iff X is algebraic variety.

$$\left\{ \begin{matrix} \text{radical ideals} \\ \text{in } \mathbb{k}[x_1, \dots x_n] \end{matrix} \right\} \underset{\mathbb{I}}{\overset{\mathbb{V}}{\rightleftarrows}} \left\{ \begin{matrix} \text{algo varieties} \\ \text{in } \mathbb{k}^n \end{matrix} \right\}$$

• **Thm**: (HN II): If $\mathbb{k}$ is algebraically closed and $F \subseteq \mathbb{k}[x_1, \dots x_n]$ and $V = \mathbb{V}(F)$ then

$$\mathbb{I}(\mathbb{V}(F)) = \mathbb{I}(V) = \sqrt{\langle F \rangle}$$

Thm: (Hilbert's Nullstellensatz): If $\mathbb{k}$ is algebraically close then $\mathbb{V}$ and $\mathbb{I}$ give mutal inverse inclusion reversing bijections

$$\left\{ \begin{array}{c} \text{radical ideals} \\ \text{in } \mathbb{k}[x_1, \dots x_n] \end{array} \right\} \underset{\mathbb{I}}{\overset{\mathbb{V}}{\rightleftarrows}} \left\{ \begin{array}{c} \text{alg. varieties} \\ \text{in } \mathbb{k}^n \end{array} \right\}$$