PAWS 2026: QUATERNION ALGEBRAS
PROBLEM SET 2

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We hope you attended and enjoyed your first watch party, and that you are now working your way through Problem Set 1! Here are some new problems based on Lecture 2, giving you an opportunity to practice and explore the ideas from the lecture further.

We will have our first problem discussion session this week, and we are really looking forward to it! Please let us know about any typos, mistakes, or comments you have on these problems, either on Discord or during our virtual sessions. You can also use the Discord channels to discuss these problems, share ideas, and ask questions with your friends.

Throughout this Problem Set, let F be a field of characteristic different from 2.

(1) In this problem, we explore quaternionic bases and isomorphic quaternion algebras.

Let $a, b \in F^\times$. Show that the quaternion algebra $(a, b)_F$ is isomorphic to each of the following by exhibiting a new quaternionic basis satisfying relations defined by these new parameters:

(a) $(b, a)_F$;

(b) $(a, -ab)_F$;

(c) (ac^2, bd^2) for any $c, d \in F^\times$.

Make sure you carry out the computations to verify these relations!

(2) Given $u, v \in (a, b)_F$ such that $u, v \notin F$, but

$$u^2 \in F^\times, \qquad v^2 \in F^\times, \qquad \text{and} \qquad uv = -vu,$$

show that

$$\{1, u, v, uv\}$$

is a linearly independent set and hence is a quaternionic basis of $(a, b)_F$.

Hint. Recall that $\mathrm{char}(F) \neq 2$. One approach is to show first that $\{1, u\}$ is independent. Then show that $\{1, u, v\}$ is independent using $uv = -vu$ and $u^2 \in F^\times$, and finally show that $\{1, u, v, uv\}$ is independent using $u^2, v^2 \in F^\times$ and $uv = -vu$.

(3) In lecture, we showed that there is an isomorphism $(1, a)_F \simeq \mathrm{M}_2(F)$. Recall how this isomorphism is defined and verify explicitly that it is indeed an isomorphism of quaternion algebras. In particular, you should verify that the map is multiplicative!

(4) Let

$$\varphi\colon (1, a)_F \xrightarrow{\sim} \mathrm{M}_2(F)$$

be the isomorphism as in the previous exercise. We identify a nilpotent element and an idempotent element in $(1, a)_F$ via this isomorphism as follows.

(a) Find the unique element $v \in (1, a)_F$ such that

$$\varphi(v) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(b) Verify directly in $(1,a)_F$ that
$$v^2 = 0.$$
(c) Find the unique element $e \in (1,a)_F$ such that
$$\varphi(e) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
and verify directly that
$$e^2 = e.$$

(5) In this problem, we explore properties of quaternion algebras over finite fields.

Let $F = \mathbb{F}_3$. Since $F^\times = \{1,2\}$, consider the four quaternion algebras
$$(1,1)_F, \qquad (1,2)_F, \qquad (2,1)_F, \qquad (2,2)_F.$$
Recall that $(a,b)_F$ is split if and only if $b \in N_a$. (Equivalently, the algebra is split if and only if $b \in N_{F(\sqrt{a})/F}(F(\sqrt{a})^\times)$ where $N_{F(\sqrt{a})/F}(x + y\sqrt{a}) = x^2 - ay^2$.)

(a) Compute the set $\{x^2 - y^2 : x, y \in \mathbb{F}_3\}$.

(b) Compute the set $\{x^2 - 2y^2 : x, y \in \mathbb{F}_3\}$.

(c) Use your computations to determine which of the four quaternion algebras above are split.

(d) What do you observe?

(6) The norm criterion for splitting gives an interesting symmetry between the two parameters of a quaternion algebra. Let F be a field of characteristic different from 2, and let $a, b \in F^\times$. Prove that
$$b \in N_{F(\sqrt{a})/F}\big(F(\sqrt{a})^\times\big) \iff a \in N_{F(\sqrt{b})/F}\big(F(\sqrt{b})^\times\big).$$
In other words, show that $b \in N_a$ if and only if $a \in N_b$.

Hint: Use the splitting criterion for quaternion algebras and the isomorphism $(a,b)_F \cong (b,a)_F$.

(7) In this problem, we consider "the same" quaternion algebras, but defined over different fields.

(a) Give an example of fields L/K and elements $a, b, c, d \in K$ such that $(a,b)_K \ncong (c,d)_K$ but $(a,b)_L \cong (c,d)_L$, or explain why this is not possible.

(b) Give an example of fields L/K and elements $a, b, c, d \in K$ such that $(a,b)_K \cong (c,d)_K$ but $(a,b)_L \ncong (c,d)_L$, or explain why this is not possible.