PAWS 2026 QUATERNION ALGEBRAS
PROBLEM SET 1

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The problems in this set build on the material from Lecture 1. Some are extensions of exercises discussed in class, giving you an opportunity to practice and explore the ideas further. Others are intended to give you a glimpse of what lies ahead and to highlight some of the broader ideas that will play a role in the course.

As you know, we will have a problem discussion session where we will work through these problems together. We encourage you to work on the problems yourself beforehand so that you can discuss your approaches and solutions—and any questions or interesting observations—with your friends during the session.

(1) Recall that for a ring R, the *center* of R is defined as
$$Z(R) = \{r \in R : rr' = r'r \text{ for all } r' \in R\}.$$
  (a) Show the center of $\mathbb{H}$ is $\mathbb{R}$, i.e.,
$$\{q \in \mathbb{H} : qq' = q'q \text{ for all } q' \in \mathbb{H}\} = \mathbb{R}.$$
  (b) Show that the center of a division ring is a field. (The main point is to show the inverse of a nonzero element in the center is also in the center.)

(2) Verify the following properties of conjugation in $\mathbb{H}$:
$$\overline{q_1 + q_2} = \overline{q_1} + \overline{q_2}, \qquad \overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}, \qquad \overline{\overline{q}} = q,$$
$$\overline{cq} = c\,\overline{q} \quad \text{for } c \in \mathbb{R}, \qquad \overline{q} = q \iff q \in \mathbb{R}.$$

(3) In this exercise, we explore how the norm is related to the conjugation and, ultimately, to invertibility in a quaternion algebra.

Let $(a, b)_F$ be a quaternion algebra over F, and let
$$v = v_0 + v_1 i + v_2 j + v_3 ij \in (a, b)_F.$$
Recall that
$$\overline{v} = v_0 - v_1 i - v_2 j - v_3 ij \qquad \text{and} \qquad N(v) = v\overline{v}.$$
  (a) Verify directly that
$$v\overline{v} = v_0^2 - av_1^2 - bv_2^2 + abv_3^2.$$
  (b) Show that
$$\overline{v}\, v = N(v).$$
  (c) Show that if $N(v) \neq 0$, then v is invertible, and find an explicit formula for v^{-1} in terms of $\overline{v}$ and $N(v)$.

(4) Let
$$A = (2, 3)_{\mathbb{Q}}.$$
Recall that A has a basis $\{1, i, j, ij\}$ over $\mathbb{Q}$, with
$$i^2 = 2, \qquad j^2 = 3, \qquad ij = -ji.$$
Let $k = ij$.
  (a) Compute
$$k^2, \qquad ik, \qquad ki, \qquad jk, \qquad kj,$$
writing each element in the basis $\{1, i, j, k\}$.

(b) Let

$$q = 1 + i + j, \qquad r = 1 - i + j.$$

Compute qr and rq.

(c) Compute $(i+j)^2$, and compare your answer with

$$i^2 + 2ij + j^2.$$

Explain why the usual commutative binomial formula does not apply.

(5) Let $a, b \in \mathbb{R}^\times$, and consider

$$A = (a, b)_{\mathbb{R}}.$$

(a) Suppose that $a < 0$ and $b < 0$. Use the norm formula to show that

$$N(q) > 0$$

for every nonzero $q \in A$. Conclude that A is a division algebra.

(b) Suppose that $a > 0$. Show that

$$(\sqrt{a} + i)(\sqrt{a} - i) = 0,$$

although neither factor is zero. Give a similar construction when $b > 0$.

(c) Conclude that $(a, b)_{\mathbb{R}}$ is a division algebra if and only if

$$a < 0 \qquad \text{and} \qquad b < 0.$$

(6) Let F be a field of characteristic different from 2, and let $a, b, c \in F^\times$. In this exercise, we show that

$$(a, b)_F \cong (ac^2, bc^2)_F.$$

Thus the parameters a and b in the presentation of a quaternion algebra are not unique.

Let i, j denote the standard generators of $(a, b)_F$ and i', j' the standard generators of $(ac^2, bc^2)_F$.

(a) Show that the elements ci, cj in $(a, b)_F$ satisfy

$$(ci)^2 = ac^2, \qquad (cj)^2 = bc^2, \qquad (ci)(cj) = -(cj)(ci).$$

(b) Use this to construct an F-algebra homomorphism

$$(ac^2, bc^2)_F \longrightarrow (a, b)_F.$$

(c) Show that this homomorphism is an isomorphism.

(7) Let F be a field of characteristic not 2, and let $A = (a, b)_F$. In this exercise, we show that A is *simple*, i.e., it has no nontrivial proper two-sided ideals. Let $k = ij$.

(a) Let I be a two-sided ideal of A; we wish to show $1 \in I$. Fix nonzero $q = q_0 + q_1 i + q_2 j + q_3 k$ with each $q_i \in F$, and suppose $q \in I$. Compute qi and iq, which are by assumption both elements of I, since I is a two-sided ideal.

(b) Since $\mathrm{char}(F) \neq 2$, we have that

$$\frac{1}{2}(iq + qi) \text{ and } \frac{j}{2}(iq - qi)$$

are elements of I. Compute both of these and show that at least one is nonzero, and therefore there exists a nonzero element $q' \in I$ of the form $q' = q_0' + q_1' i$ with each $q_i' \in F$.

(c) Compute $q'j$ and jq', which are by assumption both elements of I.

(d) Since $\mathrm{char}(F) \neq 2$, we have that

$$\frac{j}{2}(q'j + jq') \text{ and } \frac{k}{2}(q'j - jq')$$

are elements of I. Compute both of these and show that at least one is nonzero, and therefore there exists an element of F in I. Conclude that $1 \in I$, and therefore $I = A$ and A is simple.

(8) In this exercise, we show that the Hamilton quaternions can be viewed as a subring of $\mathrm{M}_2(\mathbb{C})$, and conjugation and taking norm correspond to certain matrix operations.

(a) Show that $\mathbb{H}$ can be viewed as a right $\mathbb{C}$-vector space $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$ with basis $\{1, j\}$.

(b) For any $q \in \mathbb{H}$, let
$$\ell_q : \mathbb{H} \to \mathbb{H}$$
be the map given by
$$x \mapsto qx.$$
Show that this is a $\mathbb{C}$-linear map where $\mathbb{H}$ is viewed as a right $\mathbb{C}$-vector space as above.

(c) For each of the elements $q = 1, i, j, k \in \mathbb{H}$, write down a 2×2 matrix in $\mathrm{M}_2(\mathbb{C})$ representing the linear maps ℓ_q. In other words, for each q above, find $A_q \in \mathrm{M}_2(\mathbb{C})$, such that given an element $x = x_1 + jx_2 \in \mathbb{H}$ where $x_1, x_2 \in \mathbb{C}$, we have
$$\ell_q(x) = x_1' + jx_2'$$
with
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = A_q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(d) Given a general element $q = z + jw \in \mathbb{H}$ where $z, w \in \mathbb{C}$, can you write down the matrix A_q representing ℓ_q?

(e) Show that $q \mapsto A_q$ defines an injective ring homomorphism
$$\Phi : \mathbb{H} \to \mathrm{M}_2(\mathbb{C}).$$
Note that this means you need to show that this map is both a ring homomorphism and injective!

(f) Can you describe the image of Φ?
Hint: What do you notice about the diagonal entries and the off-diagonal entries of the matrix A_q respectively?

(g) Let $\bar{q}$ denote the conjugate of $q \in \mathbb{H}$. Show
$$A_{\overline{q}} = \overline{A_q}^t \text{ (the conjugate transpose of } A_q).$$

(h) Show
$$N(q) = \det(A_q).$$