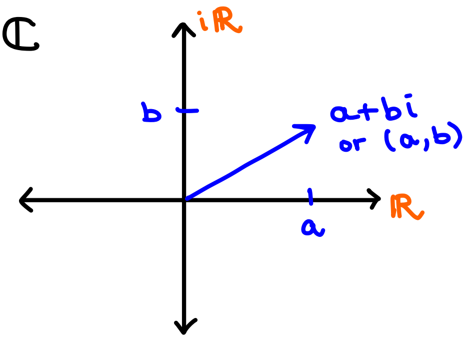
Lecture 1

From complex numbers to quaternions:

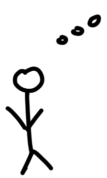


complex numbers form a 2-dim'l vector space over $\mathbb{R}$ w/ basis $\{1, i\}$

$\mathbb{C}$ also has ring structure that's compatible w/ vect. space structure. Multiplication is def. by:

$$(a+bi)(c+di) = (ad-bc) + (ac+bd)i$$

Can we multiply triples of real numbers (a,b,c) in a way that extends multiplication of complex numbers (a,b) when thought of as triples $(a,b,0)$??



Hamilton

artist's rendition

Def: The Hamilton quaternions are

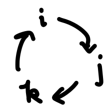
$$\mathbb{H} := \{ a + bi + cj + dk : a, b, c, d \in \mathbb{R} \}$$

with multiplication defined by the relations

$$i^2 = -1 = j^2 = k^2, \quad ij = k = -ji, \text{ and}$$

every $a \in \mathbb{R}$ commutes w/ i, j, k

Note: This ring is NOT commutative!



Ex:
$$\begin{aligned}(3i+j)(2i-j) &= 6i^2 - 3ij + 2ji - j^2 \\ &= -6 - 3ij - 2ij + 1 \\ &= -5 - 5ij \\ &= -5 - 5k\end{aligned}$$

Let's look at starting Exercise 1.4. Join watch party for additional practice:

The center of $\mathbb{H}$ is $Z(\mathbb{H}) = \{ q \in \mathbb{H} : qx = xq \ \forall x \in \mathbb{H} \}$.
$\mathbb{R} \subseteq Z(\mathbb{H})$ so exercise asks you to show $\supseteq$.

What commutes w/ i?

$$\begin{aligned}(a+bi+cj+dk)i &= ai + bi^2 + cji + dki \\ &= -b + ai + dj - ck\end{aligned}$$

vs.

$$\begin{aligned}i(a+bi+cj+dk) &= ai + bi^2 + cij + dik \\ &= -b + ai - dj + ck\end{aligned}$$

$c = d = 0$

Compare w/ elts that commute w/ j!

General Quaternion Algebras

Def: Let F (char$(F) \neq 2$) be a field and $a, b \in F^\times$. The <u>quaternion algebra</u> $(a,b)_F$ is the 4-dim'l F-algebra w/ basis $\{1, i, j, ij\}$ and w/ multiplication def. by the relations

(*) $i^2 = a$, $j^2 = b$, and $ij = -ji$.

Elements look like $q = v_0 + v_1 i + v_2 j + v_3 ij$ w/ each $v_i \in F$.

<u>Ex:</u> $\mathbb{H}$ is a quat. alg. w/ $F = \mathbb{R}$, $a = b = -1$.

★ How should you think about this? $(a,b)_F$ is a vect. space over F w/ bonus ring structure.

scalar mult by elts of F and ring mult are compatible

$$(cx)(dy) = cd(xy) \quad \forall c, d \in F$$
$$\forall x, y \in (a,b)_F$$

Keeping track of ring structure is very similar to $\mathbb{H}$: need dist. law and the relations in (*)

<u>But</u> we want to remember that there's a vector space w/ a basis here. Think back to $\mathbb{C}$. It's a (commutative) ring, but it's also a 2-dim'l $\mathbb{R}$-vect. space w/ basis $\{1, i\}$. So it's a 2-dim'l $\mathbb{R}$-algebra.

Examples (Quaternion algebras over $\mathbb{Q}$):

By varying the parameters $a, b \in \mathbb{Q}^\times$, we get different 4-diml algebras over $\mathbb{Q}$.

- In $(-1,-1)_\mathbb{Q}$, we have $i^2 = -1$, $j^2 = -1$. This is the Hamilton quaternions restricted to rational coordinates, i.e.,
  $$(-1,-1)_\mathbb{Q} = \{a + bi + cj + dk : a,b,c,d \in \mathbb{Q}\} \text{ w/ } i^2 = j^2 = k^2 = -1.$$
- In $(2,3)_\mathbb{Q}$, we have $i^2 = 2$, $j^2 = 3$, and $ij = -ji$. Note that, e.g.,
  $$(ij)(ij) = (ij)(-ji) = -i(j^2)i = -i(3)i = -3i^2 = -3(2) = -6$$

How can we access structural properties?

Analogy w/ $\mathbb{C}$: In $\mathbb{C}$, an elt. $z = x + iy$ has complex conj. $\overline{z} = x - iy$. Recall that $|z| = z\overline{z} = x^2 + y^2 \in \mathbb{R}$ and measures the squared magnitude of z.

Can generalize to quat. algs:

Def: Let $v = v_0 + v_1 i + v_2 j + v_3 ij$ be an elt. in $(a,b)_F$. The conjugate of v is

$$\overline{v} := v_0 - v_1 i - v_2 j - v_3 ij$$

The norm of v is

$$N(v) := v\overline{v} = v_0^2 - a v_1^2 - b v_2^2 + ab v_3^2$$

It turns out that the norm gives us a lot of insight about the structure of a quaternion algebra.

Prop: An element q of the quaternion algebra $(a,b)_F$ is invertible iff it has nonzero norm. In particular, $(a,b)_F$ is a division algebra iff $N : (a,b)_F \to F$ does not vanish except at 0.

Proof: The norm is a multiplicative function since

$$\begin{aligned} N(q_1 q_2) &= q_1 q_2 \overline{q_1 q_2} \\ &= q_1 q_2 \overline{q_2}\, \overline{q_1} \\ &= q_1 \underbrace{N(q_2)}_{\in F} \overline{q_1} \\ &= q_1 \overline{q_1} N(q_2) \\ &= N(q_1) N(q_2) \end{aligned}$$

check this!

If $qq' = 1$ for some elt. q', then taking norms of both sides gives:

$$N(q)N(q') = N(qq') = N(1) = 1 \in F$$

Conversely, if $N(q) \neq 0$ then

$$\frac{\overline{q}}{N(q)} =: q' \text{ is a 2-sided mult. inverse of } q. \qquad \blacksquare$$

Can the norm actually vanish?

Example: Let $A = (1,-1)_{\mathbb{Q}}$, w/ $i^2 = 1$, $j^2 = -1$, $ij = -ji$.

Consider $q = 1 + i \in A$.

$$N(1+i) = q\bar{q} = (1+i)(1-i) = 1 - i + i - i^2 = 1 - 1 = 0$$

So $(1,-1)_{\mathbb{Q}}$ is NOT a division algebra!

Can you find other examples?

Exercise 1.12 (ii): $(2,2)_{\mathbb{Q}} =: A$ is not a division alg:

If $q \in A$, then $N(q) = v_0^2 - 2v_1^2 - 2v_2^2 + 4v_3^2$.

So, e.g. taking $v_0 = 0$, $v_1 = v_2 = v_3 = 1$ gives

$q = i + j + ij$ which has norm 0! (Can check this ~~by~~ hand!)

Next time: How to tell whether two quaternion algebras are isomorphic over F