MATROIDS AND RETRACT IRRATIONALITY OF CUBIC THREEFOLDS

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Abstract. This is a brief description of the lecture course "Matroids and retract irrationality of cubic threefolds" to be held at the Arizona Winter School in March 2027.

1. Introduction

A variety X over a field k is called *retract rational* if there is an integer N and a unirational parametrization $g\colon \mathbb{P}^N_k \dashrightarrow X$, represented by a morphism $V \to U$ between dense open subsets $V \subset \mathbb{P}^N_k$ and $U \subset X$, such that for all field extensions L/k the induced map $g\colon V(L) \to U(L)$ on L-rational points is surjective. For instance, if $X = \{F = 0\} \subset \mathbb{P}^{n+1}_k$ is a hypersurface defined by a homogeneous polynomial $F \in k[x_0, \dots, x_{n+1}]$, then this means that there are rational functions $g_0, \dots, g_{n+1} \in k(t_1, \dots, t_N)$ such that the tuple

$$[g_0(t_1, \dots, t_N) : \cdots : g_{n+1}(t_1, \dots, t_N)]$$

parametrizes, for each field extension L/k, all L-rational points of a dense open subset $U \subset X$.

By a recent theorem of Banecki [Ban24], smooth retract rational varieties over infinite fields are uniformly retract rational. This means that for any given point $x \in X$, there are open subsets U and V as above such that $x \in U$. In particular, a smooth retract rational variety X over an infinite field k has the property that there are finitely many unirational parametrizations $g_i\colon \mathbb{P}^{N_i}_k \dashrightarrow X$ such that for *all* field extensions L/k, *every* L-rational point of X is parametrized by at least one of the g_i. In other words, the set of L-rational points of X can be parametrized by finitely many tuples of rational functions defined over k, uniformly in L.

Stably rational varieties are retract rational; the converse fails over some non-closed fields, but the question is open over algebraically closed fields. By [BCTSSD85], there are stably rational (hence retract rational) varieties over the complex numbers that are not rational. Further examples have recently been produced in [TZ26], where Tschinkel and Zhang show that there are smooth cubic threefolds over $\mathbb{Q}$ that are stably rational over $\mathbb{Q}$, and hence over $\mathbb{C}$. The fact that smooth cubic threefolds are never rational over a field of characteristic zero is due to Clemens and Griffiths [CG72].

Thus, some smooth complex cubic threefolds are retract rational. The goal of this course will be to explain the following theorem from joint work with Philip Engel and Olivier de Gaay Fortman, which produces the first

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examples of smooth cubic hypersurfaces of dimension at least 2 over algebraically closed fields that are *not* retract rational.

Theorem 1.1 ([EGFS25b]). *Let $Y \subset \mathbb{P}^4_{\mathbb{C}}$ be a very general cubic threefold. Then Y is not retract rational.*

The proof of the above theorem uses methods from the following three subjects, which we will touch upon in our course:

- Chow groups and decompositions of the diagonal [Ful98, Voi19];
- the Mumford construction of degenerating abelian varieties [EGFS25a];
- the combinatorial theory of regular matroids [Oxl11].

2. Plan

I plan to cover the following material in my course.

- The notion of retract rationality and how to obstruct it. Decompositions of the diagonal.
- Voisin's theorem [Voi17], stating that a smooth complex cubic threefold has a decomposition of the diagonal if and only if the minimal curve class on its intermediate Jacobian is algebraic. By [BGF23], the latter condition is equivalent to the integral Hodge conjecture for curve classes on the intermediate Jacobian.
- Regular matroids. Vanishing cycles. Families of abelian varieties associated to regular matroids via the Mumford construction [EGFS25a].
- Statement of the main result of [EGFS25b] and sketch of its proof.

To prepare for the class, it will be beneficial to read [Sch25, Section 2 and 5] in advance.

3. Problems

I will offer

- problems related to the integral Hodge conjecture and relations to rationality over the complex numbers,
- some combinatorial problems about regular matroids, and
- arithmetic problems about retract rationality for varieties over nonclosed fields.

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