THE INTERMEDIATE JACOBIAN TORSOR OBSTRUCTION TO RATIONALITY

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1. Course Introduction

Let k be a field and let X be a nice (i.e, smooth projective, geometrically irreducible) n-dimensional variety defined over k. The variety X is called rational if there exists an open dense subset $U \subset X$, which is isomorphic to an open $U' \subset \mathbb{P}^n_k$. In other words, there is a birational map $\mathbb{P}^n \dashrightarrow X$ defined over k, which we call a rational parameterization. Rational varieties are, therefore, those with simple geometry close to that of projective space. From an arithmetic geometer's perspective, it is also easy to imagine why it is nice to know that a variety is rational; for example, such varieties over $\mathbb{Q}$ have Zariski dense $\mathbb{Q}$-points, and explicit rational parameterizations can be utilized to obtain asymptotics. In another direction, if the quotient $\mathbb{A}^n_{\mathbb{Q}}/G$ of a linear action of a finite group G is rational, then Hilbert's irreducibility theorem implies that the inverse Galois problem for G has a positive answer.

Despite the simplicity of the notion of rationality, determining whether a variety is rational can be thorny; simple-to-state problems remain open for many years. In order to show that a variety X is *not* rational, you must exhibit some property of projective space that is preserved under birational maps that X does not have. This is a so-called rationality obstruction. On the other hand, if all of the rationality obstructions you've tried have vanished, that doesn't mean that your variety is rational. To prove rationality, you must provide a rationality construction, that is, a rational parameterization.

Working over a nonalgebraically closed field adds additional challenges, but also additional opportunities. On the one hand, to be rational (over k), the rational parameterization must be *defined over the ground field*, and consequently, a variety can be geometrically rational[1], but fail to be rational. On the other hand, the obstructions can take into account arithmetic information about $\mathbb{P}^n_k$ and birational maps over k. An example is the Lang–Nishimura obstruction: if X is rational, then $X(k) \neq \emptyset$.

This lecture series will focus on a particular rationality obstruction for geometrically rational threefolds, whose origin is in the classic work of Clemens–Griffiths [CG72] on the Lüroth problem: the intermediate Jacobian torsor (IJT) obstruction. This obstruction, developed by Hassett–Tschinkel when $k = \mathbb{R}$ [HT21b] and Benoist–Wittenberg in general [BW23] (see also [HT21a] for the case $k \subset \mathbb{C}$), has been powerfully applied in recent years to determine the rationality of all of the Fano threefolds of Picard rank 1 [KP23], as well as other natural cases [FJS+24, Bro, JJ24, JS25].

The intermediate Jacobian of a complex rationally connected threefold X is the complex torus

$$J^3(X) := H^{1,2}(X, \mathbb{C}) / \operatorname{Im}(H^3(X, \mathbb{Z})),$$

[1] A variety X/k is called geometrically rational if the basechange to the algebraic closure $X_{\bar{k}}$ is rational (over $\bar{k}$).

which is naturally equipped with a principal polarization, making it into a principally polarized abelian variety (p.p.a.v.). This is useful when considering rationality for the following two reasons:

- (Weak factorization) A birational map $\mathbb{P}^3 \dashrightarrow X$ can be resolved by a sequence of blowups along points and smooth curves.
- The cohomology of a variety changes in a controlled way under blowup.

This leads to the Clemens–Griffiths obstruction:

Theorem 1.1 (Clemens–Griffiths [CG72]). *Let $X/\mathbb{C}$ be a rational threefold. Then J^3_X is isomorphic as a p.p.a.v. to a product of Jacobians of smooth curves.*

If X is a geometrically rational threefold defined over an arbitrary field k, then there is a general construction of the intermediate Jacobian J^3_X of X due to Benoist–Wittenberg [BW23] (building on previous work [Del72, Mur72, Mur85, ACMV17, ACMV20]), which is a principally polarized abelian variety (and is a descent of the complex abelian variety defined above via Hodge theory when $k \subset \mathbb{C}$). Again in this case, if X is rational, then J^3_X must be isomorphic as a p.p.a.v. to a product of Jacobians of smooth curves. This was used by Benoist–Wittenberg to exhibit examples of $\mathbb{C}$-rational, $\mathbb{R}$-irrational threefolds $X/\mathbb{R}$ [BW20].

But there is more! Just as the Jacobian of a curve is the identity component of the Picard scheme, Benoist–Wittenberg's construction of the intermediate Jacobian J^3_X is as the identity component $(\mathbf{CH}^2_{X/k})^0$ of a codimension 2 Chow scheme $\mathbf{CH}^2_{X/k}$. The components of $\mathbf{CH}^2_{X/k}$ are indexed by Galois-invariant curve classes $\gamma \in \mathrm{NS}^2(X_{\bar{k}})^{G_k}$. These components $(\mathbf{CH}^2_{X/k})^\gamma$ are torsors under the identity component J^3_X, and they provide a refinement of the Clemens–Griffiths obstruction over nonclosed fields. For example:

Theorem 1.2 (Benoist–Wittenberg [BW23] (see also Hassett–Tschinkel for $k \subset \mathbb{C}$ [HT21a])). *Let X/k be a geometrically rational threefold and assume that $J^3_X \simeq \mathbf{Pic}^0_{\Gamma/k}$ for a smooth projective curve Γ of genus $g \geq 2$. Then for all $\gamma \in \mathrm{NS}^2(X_{\bar{k}})^{G_k}$, there exists $d \in \mathbb{Z}$ such that the torsor $(\mathbf{CH}^2_{X/k})^\gamma$ is isomorphic to the torsor $\mathbf{Pic}^d_{\Gamma/k}$.*

In this lecture series, we will explain the construction of the codimension 2 Chow scheme and how the intermediate Jacobian torsors give an obstruction to rationality. We will use concrete examples to demonstrate the abstract theory.

2. Course outline

(1) **Lecture 1: Rationality and irrationality over non-closed fields**. The goal of this lecture is to introduce the two aspects underlying the rationality problem (obstructions and constructions), with attention to varieties defined over fields that are not algebraically closed. Topics include the Lang–Nishimura obstruction, the real topological obstruction, and the rationality problem for quadric hypersurfaces.
(2) **Lecture 2: The classical intermediate Jacobian obstruction to rationality for threefolds**. This lecture will introduce the intermediate Jacobian (in analogy with the familiar case of the Jacobian of a curve) and Clemens–Griffiths's intermediate Jacobian obstruction to rationality over the complex numbers.
(3) **Lecture 3: The intermediate Jacobian torsor (IJT) obstruction**. In this lecture, we will upgrade the Clemens–Griffiths obstruction to take into account torsors under the intermediate Jacobian, leading to the IJT obstruction. As an application,

we will understand the rationality problem for smooth complete intersections of two quadrics in $\mathbb{P}^5$ (following work of Hassett–Tschinkel and Benoist–Wittenberg).

(4) **Lecture 4: the IJT obstruction for conic bundles**. In this final lecture, we will compute the intermediate Jacobian torsors in the case of conic bundle threefolds, following Frei–Ji–Sankar–Viray–Vogt. This case illustrates both the power, and the shortcomings, of the IJT obstruction.

3. Projects

The projects for this course will involve computing the intermediate Jacobian torsors for some natural classes of geometrically rational threefolds (including examples of conic bundles and del Pezzo fibrations), in order to explore their rationality over nonclosed fields. Corey Brooke will be the project assistant.

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