# RATIONALITY AND SPECIALIZATION: COURSE AND PROJECT OUTLINE ARIZONA WINTER SCHOOL 2027

ASHER AUEL

## Introduction

These lectures will provide an overview of the powerful specialization method that has transformed the landscape of the rationality problem in algebraic geometry over the past fifteen years. This method, which is due to Voisin [38], further developed by Colliot-Thélène and Pirutka [11], refined by Schreieder [30, 31, 33], and transformed by Nicaise, Shinder, and Ottem [24, 25], mixes powerful intuition from Hodge theory with techniques from algebraic $K$-theory, algebraic cycles, resolution of singularities, quadratic forms, Brauer groups, unramified cohomology, and (more recently) tropical geometry, and has led to breakthroughs in our understanding of the rationality of many classes of rationally connected varieties (such as hypersurfaces, complete intersections, conic and quadric bundles, and other Mori fiber spaces) as well as the deformation properties of rationality in families.

## 1. Suggested background and further reading

- **Rationality.** There are now a number of good survey articles on the history of the rationality problem, viewed from various perspectives, and how the recent advances, such as the specialization method, fit in to the story.
  - Beauville [6] writes a beautiful and historically focused survey on the rationality problem with lots of examples, concluding with a brief discussion of Voisin's original work on the specialization method;
  - Debarre [13] gives a survey "for nonspecialists" that describes lots of interesting examples and many more of the results obtained via the specialization method;
  - Voisin [39, 40] orients her survey around complex geometry and algebraic cycles;
  - Pirutka [29] gives an introduction to the specialization method and provides a computational approach to computing Brauer groups of quadric bundles;
  - Colliot-Thélène [10] has a focus on the Brauer group, unramified cohomology, and how quadrics over an arbitrary field or discrete valuation ring play a role in studying the rationality of quadric bundles, see also [12, Chapter 11];
  - Auel–Bernardara [2] contains introductory material on the required Chow theoretic and unramified cohomology background needed (though mixed in with connections to the derived category, which will not play a role in this course);
  - Peyre [28] gives a guide to the historical development of the specialization method, as well as a review of some of the landmark results;
  - Schreieder [35] recently spoke at the ICM 2026 about the rationality problem for hypersurfaces, surveying the incredible range of techniques that have gone into the problem, and specifically on the important developments to the specialization method that are required to establish the strongest results.

  For initial preparation, students may read Debarre [13, Sections 1–3, 7], Pirutka [29], and cross-reference with Voisin [39, Sections 1–3]. Reading the full bibliography in advance is not expected!

- **Algebraic cycles**, specifically the Chow group of 0-cycles and decompositions of the diagonal, play an important role in the specialization method. The classic article where (rational) decompositions of the diagonal were introduced to connect Chow groups to the geometric coniveau, with its implications on cohomology, is by Bloch and Srinivas [7]. There are many modern developments to this idea that are utilized in the specialization method, and having basic literacy in algebraic cycles (i.e., Chow groups, intersection product, correspondences) will be necessary. Some relevant background is contained in the surveys [2, Sections 1, 3.6, 3.6, 5] and in [13, Section 7], though there is probably no better place to go than Voisin's book [40] if you are looking for the full treatment with all the details.

- **Unramified cohomology**, in particular the Brauer group, has been a crucial ingredient in the rationality problem (ever since the landmark paper by Artin–Mumford [1]) and specifically in the specialization method. The canonical survey article on unramified cohomology is by Colliot-Thélène [9], and it remains an excellent reference. Schreieder [32] has a wonderful and updated survey on various generalizations of the unramified cohomology setup that are important to his refinements of the specialization method. Some of the surveys [2, 29, 10] also contain brief background material on unramified cohomology as it relates to the rationality problem. The book Gille–Szamuely [14] is a great place to learn the background material in Galois cohomology and classical Brauer groups topics.

- **Tropical.** (Optional) If you are interested in learning about the tropical aspects of the specialization method, the articles by Nicaise–Shinder [24] and Nicaise–Ottem [25] lay the foundations of how motivic volume, together with toric and tropical techniques, enters the picture and introduces a wealth of combinatorial structure. This theory is further developed and utilized in [27, 23, 41]. There is also a cycle-theoretic analogue [34] of the motivic volume approach.

## 2. Course outline

(1) Introduction to (stable) rationality problem in algebraic geometry. Unramified cohomology and the Brauer group, and their historical role in the rationality problem.

(2) Introduction to the notion of universal $\mathrm{CH}_0$-triviality and its consequences for (unramified) cohomology. First introduction to the specialization method and analysis of the special fiber, universally $\mathrm{CH}_0$-trivial resolutions, and first applications to obstructing stable rationality.

(3) Refinements on the specialization method and more applications to obstructing stable rationality as well as the behavior of stable rationality in families.

(4) The specialization method for fibrations. Applications to quadric bundles, cubic surface bundles, and other del Pezzo surface bundles.

## 3. Projects

The projects will investigate applications of the specialization method to stable rationality problems for new families of varieties, including fibrations over $\mathbb{P}^2$ and complete intersections in homogeneous varieties. Depending on the project, students will construct suitable degenerations and explicitly compute Brauer classes or other unramified (or combinatorial) invariants of singular special fibers.

## References


[1] M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. **25** (1972), no. 3, 75–95.

[2] A. Auel and M. Bernardara, *Cycles, Derived Categories, and Rationality*, Surveys on Recent Developments in Algebraic Geometry, Proceedings of Symposia in Pure Mathematics, vol. 95, AMS, 2017, pp. 199–266.

[3] A. Auel, C. Böhning, H.-C. Graf von Bothmer, and A. Pirutka, *Conic bundle fourfolds with nontrivial unramified Brauer group*, Journal of Algebraic Geometry **29** (2020), no. 2, 285–327.

[4] A. Auel, C. Böhning, and A. Pirutka, *Stable rationality of quadric and cubic surface bundle fourfolds*, European Journal of Mathematics **4** (2018), no. 3, 732–760.

[5] A. Auel, J.-L. Colliot-Thélène, and R. Parimala, *Universal unramified cohomology of cubic fourfolds containing a plane*, Brauer Groups and Obstruction Problems: Moduli Spaces and Arithmetic, pp. 29–56, Progress in Mathematics, vol. 320, Birkhäuser Basel, 2017.

[6] A. Beauville, *The Lüroth problem*, Rationality Problems in Algebraic Geometry, Lecture Notes in Math., vol. 2172, Springer, 2016, pp. 1–27.

[7] S. Bloch and V. Srinivas, *Remarks on correspondences and algebraic cycles*, Amer. J. Math. **105** (1983), 1235–125.

[8] A. Chatzistamatiou and M. Levine, *Torsion orders of complete intersections*, Algebra & Number Theory **11** (2017), 1779–1835.

[9] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, $K$-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), pp. 1–64, Proc. Sympos. Pure Math. **58**, AMS, 1995.

[10] J.-L. Colliot-Thélène, *Introduction to work of Hassett–Pirutka–Tschinkel and Schreieder*, Birational geometry of hypersurfaces, pp. 111–125, Lect. Notes Unione Mat. Ital., vol. 26, Springer, 2019.

[11] J.-L. Colliot-Thélène and A. Pirutka, *Hypersurfaces quartiques de dimension 3 : non rationalité stable*, Ann. Sci. École Norm. Sup. **49** (2016), no. 2, 371–397.

[12] J.-L. Colliot-Thélène and A. N. Skorobogatov, *The Brauer–Grothendieck Group*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 71, Springer, 2021.

[13] O. Debarre, *On rationality problems*, EMS Surv. Math. Sci. **11** (2024), no. 1, 1–35.

[14] P. Gille and T. Szamuely, *Central Simple Algebras and Galois Cohomology*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 165, Cambridge University Press, 2017.

[15] B. Hassett, A. Pirutka, and Y. Tschinkel, *Stable rationality of quadric surface bundles over surfaces*, Acta Mathematica **220** (2018), no. 2, 341–365.

[16] B. Hassett, A. Pirutka, and Y. Tschinkel, *Intersections of three quadrics in* $\mathbb{P}^7$, Surveys on Recent Developments in Algebraic Geometry, pp. 267–274, Proc. Sympos. Pure Math., vol. 95, AMS, 2017.

[17] B. Hassett and Y. Tschinkel, *On stable rationality of Fano threefolds and del Pezzo fibrations*, Journal für die reine und angewandte Mathematik (Crelles Journal), **751** (2019), 275–287.

[18] M. Kontsevich and Yu. Tschinkel, *Specialization of birational types*, Invent. Math. **217** (2019), 415–432.

[19] A. Kresch and Y. Tschinkel, *Stable rationality of Brauer–Severi surface bundles*, Manuscripta Math. **161** (2019), no. 1-2, 1–14.

[20] J.-H. Lange and S. Schreieder, *On the rationality problem for low degree hypersurfaces*, Forum of Mathematics, Pi **13** (2025), e24.

[21] W. Li, *Degree 2 del Pezzo surface bundles and stable rationality*, arXiv:2502.20761, 2025.

[22] W. Li, *On computation of unramified cohomology over non-closed fields*, arXiv:2606.29107, 2026.

[23] S. W. Moe, *On stable rationality of polytopes*, arXiv:2311.01144, 2023.

[24] J. Nicaise and E. Shinder, *The motivic nearby fiber and degeneration of stable rationality*, Inventiones mathematicae **217** (2019), no. 2, 437–477.

[25] J. Nicaise and J. C. Ottem, *Tropical degenerations and stable rationality*, Duke Mathematical Journal **171** (2022), no. 15, 3023–3075.

[26] J. C. Ottem, *A* $(2,3)$*-complete intersection fourfold with no decomposition of the diagonal*, Mathematische Zeitschrift **302** (2022), no. 1, 567–574.

[27] J. C. Ottem, *Stable rationality of some complete intersections in the Grassmannian Gr(2,5)*, Preprint/Lecture Notes, University of Oslo (2023).

[28] E. Peyre, *Progrès en irrationalité [d'après C. Voisin, J.-L. Colliot-Thélène, B. Hassett, A. Kresch, A. Pirutka, B. Totaro, Y. Tschinkel et al.]*, Astérisque (Séminaire Bourbaki) **407** (2019), no. 1123, 91–116. on-line text

[29] A. Pirutka, *Varieties That Are Not Stably Rational, Zero-Cycles and Unramified Cohomology*, Algebraic Geometry: Salt Lake City 2015, Proceedings of Symposia in Pure Mathematics, vol. 97, American Mathematical Society, Providence, RI, 2018, pp. 459–484. on-line text
[30] S. Schreieder, *Quadric surface bundles over surfaces and stable rationality*, Algebra & Number Theory **12** (2018), no. 2, 479–490.
[31] S. Schreieder, *Stably irrational hypersurfaces of small slopes*, Journal of the American Mathematical Society **32** (2019), no. 4, 1171–1199.
[32] S. Schreieder, *Unramified cohomology, algebraic cycles and rationality*, Rationality of varieties, Progr. Math., vol. 342, Birkhäuser/Springer, Cham, 2021, pp. 345–388.
[33] S. Schreieder, *Refined unramified cohomology of schemes*, Compositio Mathematica **159** (2023), no. 5, 952–1011.
[34] N. Pavic and S. Schreieder, *The diagonal of quartic fivefolds*, Algebraic Geometry **10**(6) (2023), 754–778.
[35] S. Schreieder, *Rationality of hypersurfaces*, Proceedings of the International Congress of Mathematicians (ICM 2026), vol. 3, SIAM, 2026, pp. 637–658.
[36] B. Totaro, *Hypersurfaces that are not stably rational*, J. Amer. Math. Soc. **29** (2016), 883–891.
[37] C. Voisin, *Chow Groups, Decomposition of the Diagonal, and the Topology of Fermat Varieties*, Annals of Mathematics Studies, vol. 187, Princeton University Press, Princeton, NJ, 2014.
[38] C. Voisin, *Unirational threefolds with no universal codimension 2 cycle*, Invent. Math. **201** (2015), 207–23.
[39] C. Voisin, *Stable Birational Invariants and the Lüroth Problem*, Surveys in Differential Geometry XXI: Advances in Geometry and Mathematical Physics, International Press, 2016, pp. 313–342. on-line text
[40] C. Voisin, *Birational invariants and decomposition of the diagonal*. Birational geometry of hypersurfaces, pp. 371, Lect. Notes Unione Mat. Ital., vol. 26, Springer, Cham, 2019.
[41] T. Yoshino, *Stable rationality of hypersurfaces in schön affine varieties*, arXiv:2502.08153 (2025).

Department of Mathematics, Dartmouth College, Hanover, New Hampshire
*E-mail address*: asher.auel@dartmouth.edu