Introduction to mathematical cryptography

Lecture 2: Analysing the Discrete Logarithm Problem (DLP)

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The discrete logarithm problem (recap)

Discrete Logarithm Problem (DLP)

For $g \in \mathbb{F}_p^*$ primitive element and $A \in \mathbb{F}_p^*$,
the DLP asks to find $a \in \mathbb{Z}$ so that $\exp_g(a) = A$.
Notation: $a = \mathrm{dlog}_g(A)$.

- Security of the Diffie-Hellman key exchange protocol is based on the difficulty of solving DLP.
- Morally, the hardness of DLP depends on the order of g.
- DLP becomes easier when $ord(g)$ is composite (Pohlig-Hellman) $\Rightarrow$ we work in a prime-order subgroup $\langle g \rangle \subset \mathbb{F}_p^*$ (with g not primitive, but $ord(g) = q$ prime).

Linear search (naive algorithm)

Algorithm 0 Linear search

Input: $g \in \mathbb{F}_p^*$ with $ord(g) = q$, and $A \in \langle g \rangle$

Output: $a = \mathrm{dlog}_g(A)$

1: $x \leftarrow 1$
2: **for** $i = 1, \ldots, q$ **do**
3:     $x \leftarrow g \cdot x$
4:     **if** $A = x$ **then**
5:         **return** i
6:     **end if**
7: **end for**

Correctness

- In Line 4, at step i: $x = g^i$, so $A = x \Leftrightarrow i = \mathrm{dlog}_g(A)$.

$\Rightarrow$ Algorithm 0 terminates and the output is correct.

Runtime

(in number of multiplications M)

- Best case: $a = 1 \Rightarrow 1\,\mathrm{M}$
- Worst case: $a = q \Rightarrow q\,\mathrm{M}$
- big-O notation:[a] $O(q)$

Memory $O(1)$

[a] $f(n) = O(h(n))$ if $\exists\, c > 0, n_0 \in \mathbb{N}: |f(n)| \leq c|h(n)|\ \forall\, n > n_0$.

Baby-step giant-step algorithm

BSGS algorithm (informal)

Baby-step giant-step algorithm (BSGS) by David Shanks (1971)

Idea: decompose the solution $a = \mathrm{dlog}_g(A)$ as

$$a = jm + i, \quad \text{with } m = \lfloor \sqrt{q} \rfloor + 1,\ i, j \in \{0, \dots, m-1\}.$$

- baby steps:

$$g_0 = 1,\ g_1 = g,\ g_2 = g^2,\ \dots,\ g_{m-1} = g^{m-1}.$$

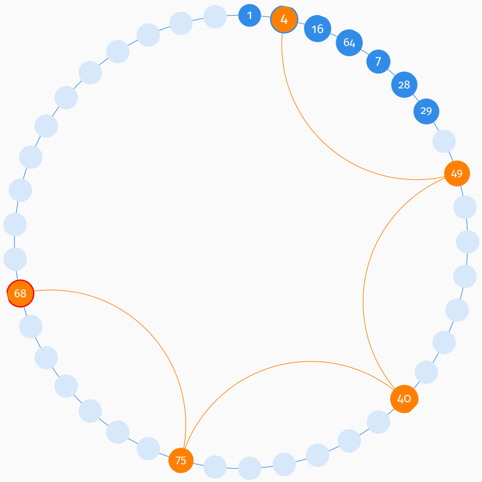
- giant steps:

$$A_0 = A,\ A_1 = g^{-m} \cdot A,\ A_2 = \left(g^{-m}\right)^2 \cdot A,\ \dots,\ \left(g^{-m}\right)^{m-1} \cdot A.$$

- A match $g_i = A_j$ provides us with the solution $a = jm + i$.

Example

Challenge find $a = \mathrm{dlog}_g(A)$ where $g = 4 \in \mathbb{F}_{83}^*$, $A = 68 \in \langle g \rangle$.



Precomputations:
$m = \lfloor \sqrt{41} \rfloor + 1 = 7$,
$h = g^{-7} = 78$.

baby steps

1, 4 (4), 16, 64, 7, 28 29.

giant steps

68, 75, 40, 49, (4), ...

Formal description

Algorithm 1 Shank's BSGS algorithm

Input: $g \in \mathbb{F}_p^*$ with $ord(g) = q$, $A \in \langle g \rangle$
Output: $a = \mathrm{dlog}_g(A)$

$m \leftarrow \lfloor \sqrt{q} \rfloor + 1$
$g_0 \leftarrow 1$
for $i = 1, \ldots, m-1$ **do**
  $g_i \leftarrow g \cdot g_{i-1}$
end for
$\tilde{A} \leftarrow A$
for $j = 0, \ldots, m-1$ **do**
  if $\tilde{A} = g_i$ for some i **then**
    return $jm + i \pmod{q}$
  else
    $\tilde{A} \leftarrow \tilde{A} \cdot g^{-m}$
  end if
end for

Correctness

✓ For $a \in \{1, \ldots, q-1\}$, $\exists\, 0 \leq i, j \leq m-1$ with $a = jm + i$.

Runtime

(in $\mathbb{F}_p$ multiplications M)

- baby steps: $(m-1)\,\mathtt{M}$
- giant steps: $\leq (m-1)\,\mathtt{M}$
- **total** $O(m) = O(\sqrt{q})$

Memory

(in $\mathbb{F}_p$ elements)

- list of baby steps: $O(\sqrt{q})$

Time-memory trade-offs

Having enough memory to store $O(\sqrt{q})$ $\mathbb{F}_p$-elements memory is unrealistic for a “real-world DLP” challenge.

⇒ **Variant of BSGS with memory restrictions**

- Available storage: $m \ll \sqrt{q}$ (finite field elements)
- baby steps: $g_0 = 1, g_1 = g, \ldots, g_{m-1} = g^{m-1}$
- giant steps: $A_0 = A, A_1 = g^{-m} \cdot A, \ldots, A_{\lfloor q/m \rfloor} = \left(g^{-m}\right)^{\lfloor q/m \rfloor}$

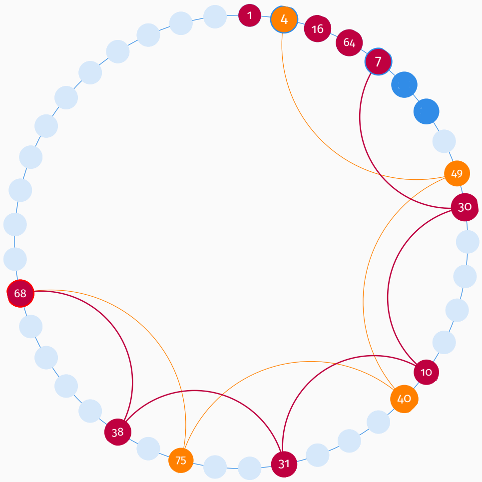
- Runtime: $O(q/m)$ $\mathbb{F}_p$-multiplications
- Memory: $O(m)$ $\mathbb{F}_p$-elements

Remarks

- If $m = \sqrt{q}$, then $O(q/m) = O(\sqrt{q})$ (as in the standard BSGS algorithm).
- *If we allow $m > \sqrt{q}$, is the above variant of BSGS faster?*
  - ⚠ The runtime $O(q/m)$ is only correct if $m \leq \sqrt{q}$.
  - A different (!) variant can provide a time-memory trade-off in the other direction (see exercises).

Example (with time-memory trade-off)

Challenge find $a = \mathrm{dlog}_g(A)$ where $g = 4 \in \mathbb{F}_{83}^*$, $A = 68 \in \langle g \rangle$.



Standard BSGS:
$m = \lfloor \sqrt{41} \rfloor + 1 = 7$,
$h = g^{-7} = 78$.

Time-memory trade-off:
$m = 5$, $h = g^{-5} = 3$

Pollard's rho algorithm

Pollard's rho algorithm (informal)

Pollard's rho algorithm suggested by John M. Pollard (1978)

Idea to find $a = \mathrm{dlog}_g(A)$.
Create a sequence

$$x_0 = g^0 A^0,\ x_1 = g^{k_1} A^{\ell_1},\ x_2 = g^{k_2} A^{\ell_2},\ \ldots$$

$\Rightarrow$ Collision in the sequence $x_i = x_j$ yields a solution:

$$g^{k_i} A^{\ell_i} = g^{k_j} A^{\ell_j} \quad \Leftrightarrow \quad A^{\ell_i - \ell_j} = g^{k_j - k_i}$$

so $a = (k_j - k_i)/(\ell_i - \ell_j)$ if $\ell_i \neq \ell_j \pmod q$.

How to construct a suitable sequence?

- Given x_i, we need to be able to compute (or already know) ℓ_i, k_i.
- The sequence should look random. Then the birthday paradox tells us that a collision occurs after $O(\sqrt{q})$ steps.

A pseudorandom sequence

Consider $f : \mathbb{F}_p^* \to \mathbb{F}_p^*$ defined by

$$x = g^k A^\ell \quad \mapsto \quad x' = \begin{cases} g^k A^{\ell+1} & \text{if } 0 < x < p/3, \\ g^{2k} A^{2\ell} & \text{if } p/3 < x < 2p/3, \\ g^{k+1} A^\ell & \text{if } 2p/3 < x < p. \end{cases}$$

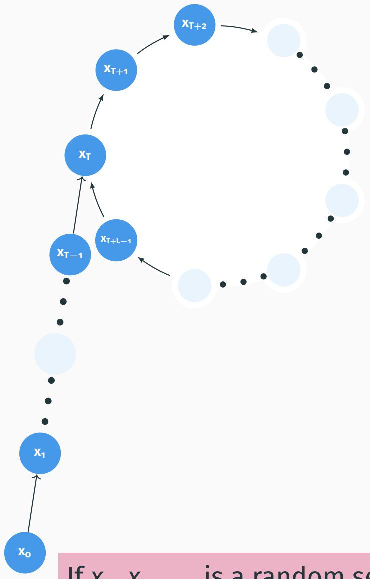
Slight abuse of notation: $f : \mathbb{F}_p^* \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \to \mathbb{F}_p^* \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$, i.e. $(x, k, \ell) \mapsto (x', k', \ell')$ to keep track of the exponents.

Pseudorandom sequence

$$x_0 = g^0 A^0,\ x_1 = f(x_0) = g^{k_1} A^{\ell_1},\ x_2 = f(x_1) = g^{k_2} A^{\ell_2},\ \ldots$$

- This sequence is periodic, since $\mathbb{F}_p^*$ is finite.
- It enters into a loop at some point.

Illustration of the Pollard's rho

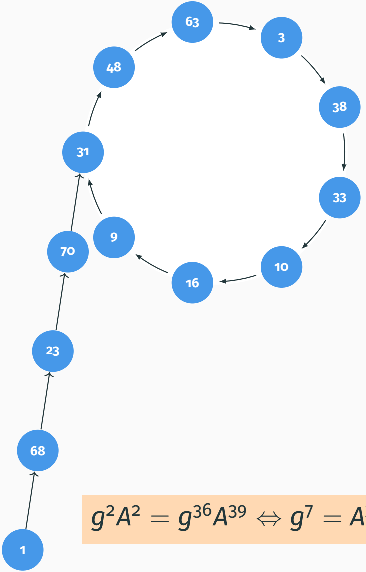


- Tail $x_0, \ldots, x_T$ of tail length T
- Loop $x_T, \ldots, x_{T+L-1}, x_{T+L} = x_T$ of loop length L
- **Birthday paradox** In a class of 30 students, what's the probability that at least two of them share the same birthday?
  $\rightarrow 1 - \prod_{i=1}^{29}(1 - \frac{i}{365}) \approx 71\%$

$\Rightarrow$ **Rule of thumb**: Drawing elements at random from a set of size N, we expect a collision after $\sqrt{\pi N/2}$ draws.

If $x_0, x_1, \ldots$ is a random sequence, we expect $T + L \approx \sqrt{\pi q/2}$.

Example of a Pollard's rho



Compute $\mathrm{dlog}_g(A)$ for $g = 4 \in \mathbb{F}_{83}^*$, $A = 68 \in \langle g \rangle$ (recall $q = 41$)

- $x_0 = g^0 A^0 = 1$
- $x_1 = g^0 A^1 = 68$
- $x_2 = g^1 A^1 = 23$

$\ldots$

- $x_4 = g^2 A^2 = 31$

$\ldots$

- $x_{13} = g^{36} A^{39} = 31$

$T = 4$ (for $x_1, \ldots, x_4$)

$L = 9$ (for $x_4, \ldots, x_{13}$)

$g^2 A^2 = g^{36} A^{39} \Leftrightarrow g^7 = A^{37}$, so $\mathrm{dlog}_g(A) = 7 \cdot 37^{-1} = 29$

Detecting collisions

Given a sequence $x_0, x_1, \ldots$ with tail length T and loop length L, find a collision $x_i = x_j$.

- **Naive strategy** Compute and store $x_0, x_1, x_2, \ldots$ until a collision occurs.
  - runtime: $L + T \approx \sqrt{\pi q/2}$ $\mathbb{F}_p^*$-multiplications
  - memory: $L + T \approx \sqrt{\pi q/2}$ $\mathbb{F}_p$-elements
- **Memory-less variant** Compute (x_i, x_{2i}) for $i = 1, 2, \ldots$ until $x_i = x_{2i}$ for some i.
  - memory: constant (sequence elements are not stored)
  - runtime: $\leq 3 \cdot (T + L) \approx 3\sqrt{\pi q/2}$ $\mathbb{F}_p^*$-multiplications

    Proof idea: We have $x_i = x_{L+i}$ for all $i \geq T$, hence

    $$x_i = x_{2i} \quad \Leftrightarrow i \equiv 2i \pmod{L} \text{ and } i \geq T.$$

    There is precisely one value $T \leq i < T + L$ satisfying these conditions.

Pollard's rho algorithm

Algorithm 2 Pollard's rho algorithm

Input: $g \in \mathbb{F}_p^*$ with $ord(g) = q$, $A \in \langle g \rangle$
Output: $a = \text{dlog}_g(A)$

```
1:  (x, k, ℓ) = (1, 0, 0)
2:  (x', k', ℓ') = (x, k, ℓ)
3:  while True do
4:      (x, k, ℓ) ← f(x, k, ℓ)                          ▷ x = x_i
5:      (x', k', ℓ') ← f(x', k', ℓ')
6:      (x', k', ℓ') ← f(x', k', ℓ')                    ▷ x' = x_{2i}
7:      if x' = x then
8:          if gcd(ℓ' − ℓ, q) = 1 then
9:              return (k − k')(ℓ' − ℓ)^{−1} (mod q)
10:         else
11:             go back to 1 (and change x_0)
12:         end if
13:     end if
14: end while
```

Correctness

✓ If the algorithm terminates (Line 9), the output is correct.

Runtime

(in $\mathbb{F}_p^*$-multiplications)

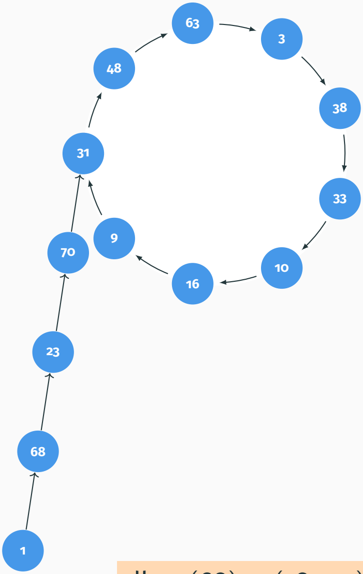
- If f is sufficiently random, then the runtime is expected to be $O(\sqrt{q})$.

Memory

(in $\mathbb{F}_p$ elements)

- only the current sequence elements (x_i, x_{2i}): $O(1)$

Example (Pollard's rho algorithm)



Compute $\mathrm{dlog}_g(A)$ for $g = 4 \in \mathbb{F}_{83}^*$, $A = 68 \in \langle g \rangle$, recall $q = 41$.
(Example from Slide 11)

- $T = 4$
- $L = 9$

$\rightarrow$ expected collision $x_i = x_{2i}$: $4 \leq i < 13$ with $i \equiv 2i \pmod 9$. $\Rightarrow i = 9$.

- We compute $(x_9, k_9, \ell_9) = (33, 18, 18)$ and $(x_{18}, k_{18}, \ell_{18}) = (33, 3, 27)$, and find

$$\mathrm{dlog}_4(68) = (18 - 3) \cdot (27 - 9)^{-1} \equiv 29 \pmod{41}.$$

Index calculus

Index calculus (informal)

Index calculus method going back to Maurice Kraitchik (1922)

Main ingredient Lift $x \in \mathbb{F}_p^*$ to $\hat{x} \in \{1, \ldots, p-1\} \subset \mathbb{Z}$

Setup (to find $a = \mathrm{dlog}_g(A)$)

- Set $\mathcal{P}_B = \{p_1, \ldots, p_b\}$ primes $p_i \leq B$ for some smoothness bound B. Denote $x_i = \mathrm{dlog}_g p_i$

Phase 1: Relation creation

- Find $b+1$ relations among $x_1, \ldots, x_b$ and a, i.e.relations of the form

$$e_1 x_1 + \ldots e_b x_b + a \equiv e \pmod{p-1}.$$

Phase 2: Linear algebra over $\mathbb{Z}/(p-1)\mathbb{Z}$.

- Solve the system of $b+1$ relations in $x_1, \ldots x_b$ and a

Phase 1: Relation creation

Goal Find $b+1$ relations among $x_1, \ldots, x_b$ and a with $x_i = \mathrm{dlog}_g(p_i)$.

Set $\mathcal{R} = \{\}$, until $\#\mathcal{R} = b+1$, repeat the following:

(1) Let $e \in \{1, \ldots, p-1\}$ random.

(2) Compute the lift $z = \widehat{g^e/A} \in \{1, \ldots, p-1\} \subset \mathbb{Z}$

(3) If z is B-smooth, i.e. if $z = \prod_{i=1}^{b} p_i^{e_i}$ for some e_i:
add R_e : $e \equiv e_1 x_1 + \cdots + e_b x_b + a$ to $\mathcal{R}$.

number of primes p_i:
$b = \pi(B) \approx B/\log(B)$
(Prime Number Theorem by Hadamard and de la Vallée Poussin, 1896)

Testing B**-smoothness**:
$O(b)$, for example using trial division[a]

Number of iterations N:
$N \approx u^u$, where $u = \log p / \log B$
(Dickman–de Bruijn function, proportion of smooth number)

[a] There are faster methods like the elliptic curve method (ECM).

Phase 2: Linear algebra, and overall runtime

Phase 2: A system of linear equations over $\mathbb{Z}/(p-1)\mathbb{Z}$

$$\begin{cases} e = e_1 x_1 + \dots e_b x_b + a \\ \qquad \vdots \\ e' = e'_1 x_1 + \dots e'_b x_b + a \end{cases}$$

can be solved in $\tilde{O}(b^3)$, or even $\tilde{O}(b^2)$ using special properties of the system.

$\Rightarrow$ time complexity is neglible compared to Phase 1

Overall complexity

- Dependence on B: time for smoothness testing increases while number of iterations decreases for increasing B.

- Optimal choice: $B \approx e^{1/2\sqrt{\log p \log \log p}}$ (in our setting)

$\Rightarrow$ Runtime: $O\left(e^{2\sqrt{\log p \log \log p}}\right)$ **subexponential**.

Example (Index calculus)

Compute $\mathrm{dlog}_g(A)$ for $g = 4 \in \mathbb{F}_{83}^*$, $A = 68 \in \langle g \rangle$, recall $q = 41$.

Factor base $\mathcal{P}_B = \{2, 3, 5\}$, i.e. $b = 3$.

Phase 1 Choose random exponents $e \in \{1, \ldots, 82\}$, and check for smoothness.

e	$\widehat{g^e/A}$	e	$\widehat{g^e/A}$
15	5^2	31	2^4
59	x	33	x
60	3^2	17	x
7	x	(72)	(2^4)
40	x	43	$2 \cdot 5$

Phase 2

$$\begin{cases} 15 = & & & 2x_5 + a \\ 60 = & & 2x_3 & + a \\ 31 = & 4x_2 & & + a \\ 43 = & x_2 & + & x_5 + a \end{cases} \Rightarrow \quad a = 29 \equiv 70 \pmod{41}$$

(here: $x_2 = \mathrm{dlog}_4(2)$, $x_3 = \mathrm{dlog}_4(3)$, $x_5 = \mathrm{dlog}_4(5)$)

Diffie-Hellman in generic groups

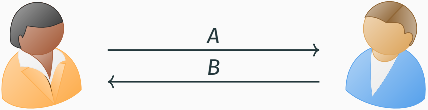
Generalization of the Diffie-Hellman protocol

Let $(\mathbb{G}, \circ)$ be a <u>commutative</u> group.

For $g \in \mathbb{G},\ a \in \mathbb{Z}$ define

$$\exp_g(a) = \underbrace{g \circ \cdots \circ g}_{a \text{ times}}.$$

Generalized Diffie-Hellman protocol



Alice	Bob
$(a \in \mathbb{Z})$	$(b \in \mathbb{Z})$
$A \leftarrow \exp_g(a)$	$B \leftarrow \exp_g(b)$
$K_{AB} \leftarrow \exp_B(a)$	$K_{BA} \leftarrow \exp_A(b)$

Group Discrete Logarithm Problem (Group-DLP)

For $g \in \mathbb{G}$, $A \in \langle g \rangle$ the <u>Group-DLP</u> asks to find $a \in \mathbb{Z}$ so that $\exp_g(a) = A$. Notation: $a = \mathrm{dlog}_g(A)$.

How hard is the Group-DLP?

Generic algorithms

- Baby-step giant-step algorithm, Pollard's rho algorithm:
  They can be applied to any group, no special property of finite fields are used.

$\Rightarrow$ Group-DLP can be solved in $O(\sqrt{N})$, where $N = \#\mathbb{G}$.
This is best possible in a generic group with N prime (Shoup, 97)

Non-generic algorithms

- Index calculus: This algorithm relies on working in finite fields (using lifts to $\mathbb{Z}$ and factorization). It cannot be translated to arbitrary groups.

Next lecture

- Elliptic curves: Groups that are closer to "generic groups"