COMPUTING MODULAR FORMS: COURSE OUTLINE AND POSSIBLE PROJECT SUMMARIES AWS 2025

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Introduction

These lectures survey complementary frameworks for computing with automorphic forms over number fields, including classical methods for modular curves (and their cousins over other fields), cohomological methods for arithmetic groups, and algebraic modular forms. A unifying theme is that systems of Hecke eigenvalues can be realized in cohomology in an algorithmically accessible way, and interpreting double cosets together with linear algebra delivers concrete data: eigenvalues, eigenvectors, Fourier expansions (when available), Galois representations, and L-functions.

1. Suggested background reading (pre-AWS)

- Basic algorithms in computational algebra and number theory form the foundation of most modern algorithms. References include Cohen [C93, C00] and von zur Gathen–Gerhard [vzGG13].
- Classical modular forms: here there are many good references, but for a first go see Voight [V21, Chapter 40] and the references listed there.
- Stein [St08] wrote a nice short survey on computing modular forms using modular symbols, which connects with the start of the lectures.
- The beginnings of linear algebraic groups (include bits of Lie theory, symmetric spaces) are helpful: a computationally-minded reference is de Graaf [deG17].

2. Course outline

(1) **Classical modular symbols and the trace formula on GL_2 over $\mathbb{Q}$.**

We set up modular symbols in homology: see Stein [St07] and Assaf [A21]. For $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup, the relative homology $H_1(X_\Gamma, \{\text{cusps}\}; \mathbb{Z})$ is generated by symbols $\{\alpha, \beta\}$ with $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ modulo the Manin relations, and carries a Hecke action via correspondences. Concretely, for T_p one uses the double coset decomposition

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma = \bigsqcup_i \Gamma \gamma_i,$$

and defines $T_p[\alpha, \beta] = \sum_i [\gamma_i \alpha, \gamma_i \beta]$ in homology. Reduction gives a canonical $\mathbb{Z}$-basis indexed by Γ-classes of unimodular column vectors, turning T_p into a reasonably sparse integer matrix. Using the Eichler–Shimura isomorphism, cusp forms $S_k(\Gamma)$ occur in H_1 with coefficients in a local system; in weight 2, we can read off eigenvalues

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directly from homology and thereby q-expansions from modular symbol data via period integrals and rational reconstruction. Complexity and sparsity estimates can be made as a function of of the level and the prime p.

We then introduce the trace formula method, as explained by Belabas–Cohen [BC18], as a "weighted class number" formula for the trace of a Hecke operator. One assembles traces of T_p and then q-expansions for newforms in a formulaic way. We explain how this complements chain-level computations, with different asymptotics.

(2) **Computing modular forms more generally.**

We then generalize to higher rank: for a start, see Gunnells [Gu07]. We begin with a reductive algebraic group $\mathbf{G}$ defined over a number field F, with real points $G_\infty = \mathbf{G}(\mathbb{R})$ and maximal compact subgroup K_∞. The associated symmetric space $D = G_\infty/K_\infty$ (when $\mathbf{G}$ is semisimple) provides the archimedean analytic input. For instance, with $\mathbf{G} = \mathrm{Sp}_{2n}$ over $\mathbb{Q}$, we recover Siegel upper half-space; for $\mathbf{G} = \mathrm{GL}_{2,F}$ we obtain a product of hyperbolic spaces $(\mathcal{H}_2)^r \times (\mathcal{H}_3)^c$ depending on the signature of F. Finite adeles $\widehat{F}$ and a compact open subgroup $\widehat{K} < \mathbf{G}(\widehat{F})$ specify the level.

The arithmetic quotient $Y = Y(\mathbf{G}, \widehat{K}) := G\backslash(D \times \widehat{G}/\widehat{K})$ decomposes into finitely many orbifolds $\Gamma_i\backslash D$ with congruence subgroups $\Gamma_i \subset \mathbf{G}(F)$. In the familiar case $\mathbf{G} = \mathrm{GL}_{2,F}$ with $\widehat{K} = \mathrm{GL}_2(\widehat{\mathbb{Z}_F})$, this quotient splits according to the narrow class group of F.

A weight is given by an algebraic representation $\rho\colon \mathbf{G} \to \mathrm{GL}(W)$ with W finite-dimensional. A modular form of weight W and level $\widehat{K}$ is then a W-valued function on $D \times \mathbf{G}(\widehat{F})/\widehat{K}$, analytic in D, locally constant adelically, and invariant under $\mathbf{G}(F)$. The resulting spaces $M_W(\mathbf{G}, \widehat{K})$ are spaces of modular forms.

The cohomological approach interprets W as defining a local system $\mathscr{W}$ on Y, with cohomology groups $H^q(Y, \mathscr{W}) \simeq \bigoplus_i H^q(\Gamma_i, W)$. By a theorem of Franke, cuspidal automorphic forms embed here. Hecke operators act through double cosets $\widehat{K}\pi\widehat{K}$, realized either via correspondences on Y or explicit coset decompositions, endowing cohomology with a rich Hecke module structure. One distinguishes cuspidal from Eisenstein cohomology, and studies base change, lifting, and oldform/newform decompositions via degeneracy maps.

This framework encompasses a wide range of groups beyond GL_2, and provides a powerful route to explicit computations: implementing Hecke actions and isolating cuspidal eigenpackets in cohomology.

(3) **Algebraic modular forms.**

Next we turn to the special case of algebraic modular forms, following Gross [Gr99]. Suppose that the real points G_∞ of the group (modulo center) is compact. Then the underlying space Y is a finite set, and so the associated space of modular forms has no geometry, coming "entirely from the finite ideles": there is no analysis at infinity. The origins of this come from the case of a totally definite quaternion algebra (originally studied over $\mathbb{Q}$ by Brandt and Eichler), where Hecke operators act on classes of right ideals, computationally realized by Brandt matrices. We explain the neighbor method and local type theory that organizes the double cosets. In the case where $\mathbf{G}$ is an orthogonal, unitary, or symplectic group and the weight corresponds to the stabilizer of a lattice, Hecke operators are built from neighbors in the genus and this provides a nice computable framework, following Greenberg–Voight [GV11].

(4) **Complementary methods.**
We conclude by surveying remaining complementary directions. For example, higher-rank modular-symbol analogues (as in sharbly complexes) provide explicit chain complexes for GL_n over number fields; Hecke acts on chains, and reduction theory yields practical finite presentations. If time permits, we also speculate on future work including trace formula algorithms.

3. Projects

(1) Compute the Hecke operator at p corresponding to the Frobenius automorphism when p divides the level of the (general congruence) subgroup Γ.
(2) Compute q-expansions of Picard modular forms similar to algorithms for Hilbert modular forms, including the graded ring.
(3) Study the modular automorphism group of a Shimura curve for a general congruence subgroup and the induced action on modular forms. Add this data to the LMFDB.
(4) Study the trace formula for reductive groups beyond $\mathrm{GL}_{2,\mathbb{Q}}$.
(5) Study explicit functoriality of lifts on algebraic modular forms.
(6) Study methods for algebraic modular forms on symplectic groups.
(7) Computing algebraic modular forms on an exceptional group, maybe G_2, using lattice methods.
(8) Compute all reduced totally positive definite binary quadratic forms up to a fixed norm discriminant over a real quadratic field, generalizing the algorithm of Gauss. Compute tables of class numbers and apply this to the trace formula for Hilbert modular forms.
(9) Use theta series or Eisenstein series to compute the Fourier expansion of a classical newform up to a q-expansion bound M in quasi-linear time in M.
(10) Can the Fourier expansion of a Siegel paramodular form for $\mathrm{GSp}_{4,\mathbb{Q}}$ be reconstructed algorithmically from its Hecke eigenvalues (i.e., its L-function) using theta series or Eisenstein series?
(11) Compute graded rings of Siegel paramodular forms in small level.
(12) Study Siegel modular forms arising from rank 6 lattices in the kernel of $\theta^{(1)}$.

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