COMPUTING MODULAR FORMS

(DAY 4)

LAST TIME:

GIVEN:
- G OVER F
- LEVEL $\hat{K} \leq \hat{G}$
- WEIGHT W
- DEGREE $r \geq 0$.

RETURN: $\{[T_{\hat{\pi}}]\}_{\hat{\pi}} \circlearrowright M_{W,r}(\hat{K})$

$$M_{W,r}(\hat{K}) \cong H^r(Y(\hat{K}), W).$$

WHERE $Y(\hat{K}) = G \backslash (D_\infty \times \hat{G}/\hat{K})$

$$Y(\hat{K}) = \bigsqcup_i Y(\Gamma_i)$$

G IS TOTALLY DEFINITE IF $D_\infty = \{\cdot\}$

$\Rightarrow Y(\hat{K}) = G \backslash \hat{G} / \hat{K}$ FINITE CLASS SET

$r = 0$

ALGEBRAIC MODULAR FORMS (GROSS)

EX: $Y(\hat{K}) = Cl\, \mathbb{Z}_F$

$= Cls\, \mathcal{O}$, $\mathcal{O}$ QUATERNION ORDER

TODAY: DEFINITE ORTHOGONAL CASE (+ OVERVIEW)

LET F BE TOTALLY REAL,

$R = \mathbb{Z}_F$, $d := [F : \mathbb{Q}]$.

$V \cong F^m$ TOTALLY DEFINITE QUADRATIC SPACE

$Q : V \to F$ EXTENDS

$$v \mid \infty \qquad V \otimes \underset{\mathbb{R}}{\underset{\wr}{F_v}} \cong \mathbb{R}^m$$

$G := O(V)$ ORTHOGONAL GROUP

$$G = O(V)(F)$$

$$= \{ g \in GL(V)(F) : Q(gv) = Q(v) \ \forall v \in V \}.$$

LET $\Lambda \subset V$ BE A (FULL) **R-LATTICE**: F.G. R-SUBMODULE WITH $F\Lambda = V$.

EX: $\Lambda = \bigoplus_{i=1}^{m} \mathbb{Z}_F v_i$, $\{v_i\}_i$ F-BASIS FOR V.

LATTICES Λ, Π ARE **ISOMETRIC** $(\Lambda \simeq \Pi)$ IF $\exists\, g \in G$ S.T. $g\Lambda = \Pi$.

$$\Lambda \rightsquigarrow Q_\Lambda := Q|_\Lambda$$

CAN WORK OVER $\mathbb{Z}$:

$$\mathrm{Tr}\,Q : \Lambda \longrightarrow \mathbb{Z}$$
$$x \mapsto \mathrm{Tr}_{F/\mathbb{Q}}\, Q(x)$$

DEFINITE QUADRATIC FORM OVER $\mathbb{Z}$,

$\mathrm{rk}_{\mathbb{Z}} \Lambda = (\mathrm{rk}_R \Lambda) \cdot [F : \mathbb{Q}] = md.$

<u>THM</u> (HAVIV–REGEV):

THE ISOMETRY PROBLEM

GIVEN: $\mathbb{Z}$-LATTICES Λ, Λ' rank n

RETURN: true OR false

ACCORDING AS $\Lambda \simeq \Lambda'$

(AND $g : \Lambda \xrightarrow[\text{ISOMETRY}]{\sim} \Lambda'$ IF true)

CAN BE SOLVED IN TIME

$n^{O(n)}$ · POLYNOMIAL IN INPUT SIZE.

IN PRACTICE, ALGORITHM PLESKEN–SOUVEIGNER.

CAN BOOTSTRAP USING ADD'L QUADRATIC FORMS TO TEST FOR ISOMETRIES OF R-LATTICES.

<u>GENUS</u>: $\mathrm{Gen}(\Lambda) = \{\text{LOCALLY ISOMETRIC LATTICES}\}$

$\hat{\Lambda} = \Lambda \otimes_R \hat{R}$

$$\begin{aligned} \mathrm{Gen}(\Lambda) &= \{\Pi \subseteq V : \hat{\Lambda} \simeq \hat{\Pi}\} \\ &\updownarrow \\ \hat{G}/\hat{K} \qquad & \hat{\Pi} = \hat{g}\hat{\Lambda} \\ & \downarrow \\ & \hat{g}\hat{K}, \quad \hat{K} := \mathrm{Stab}_{\hat{G}} \hat{\Lambda} \\ & \qquad\qquad = O(\hat{\Lambda}) \end{aligned}$$

CLASS SET:

$\mathrm{Cls}\ \Lambda$ = (GLOBAL) ISOMETRY CLASSES IN $\mathrm{Gen}(\Lambda)$

$$= G \backslash \hat{G} / \hat{K}$$

$$\{[\Lambda_i]\}_i = \coprod_i^{<\infty} G \hat{\beta}_i \hat{K}$$

$$\Lambda_i = \hat{\beta}_i \hat{\Lambda} \cap V$$

$$\Gamma_i = \hat{\beta}_i \hat{K} \hat{\beta}_i^{-1} \cap G = O(\Lambda_i)$$

$$\# \Gamma_i < \infty$$

EX: $Q = x_1^2 + x_2^2 + 3x_3^2 + 3x_4^2 + x_1 x_4 + x_2 x_3$.

$\Lambda = \mathbb{Z}^4$, GRAM MATRIX: $\begin{pmatrix} 2 & & & 1 \\ & 2 & 1 & \\ & 1 & 6 & \\ 1 & & & 6 \end{pmatrix}$

$\det = 11^2$.

$\# \mathrm{Cls}\ \Lambda = 3$

HECKE OPERATORS COME FROM NEIGHBORING LATTICES, EXHAUSTING THE CLASS SET.

INVARIANT FACTORS:
IF $\Lambda, \Pi \subset V$ ARE LATTICES, THEN THERE EXISTS AN F-BASIS OF V, $e_1, \ldots, e_m$, S.T.

$$\Lambda = \mathfrak{a}_1 e_1 \oplus \cdots \oplus \mathfrak{a}_m e_m$$

$$\Pi = \mathfrak{b}_1 e_1 \oplus \cdots \oplus \mathfrak{b}_m e_m$$

$\mathfrak{a}_i, \mathfrak{b}_j \subset F$ FRAC IDEALS.

S.T. $\mathfrak{b}_1 \mathfrak{a}_1^{-1} \supseteq \cdots \supseteq \mathfrak{b}_m \mathfrak{a}_m^{-1}$

INVARIANT FACTORS, UNIQUELY DETERMINED

DEF: Π IS A **$\mathfrak{p}, k$-NEIGHBOR** OF Λ $(\Pi \sim_{\mathfrak{p},k} \Lambda)$ IF INVARIANT FACTORS

$$\underbrace{\mathfrak{p}^{-1},\dots,\mathfrak{p}^{-1}}_{k}, R,\dots, R, \underbrace{\mathfrak{p},\dots,\mathfrak{p}}_{k}$$

$$\Longleftrightarrow \quad \begin{tikzcd}[column sep=small] \Lambda \arrow[dr, dash, "(R/\mathfrak{p})^k"'] & & \Pi \arrow[dl, dash, "(R/\mathfrak{p})^k"] \\ & \Lambda \cap \Pi \arrow[d, dash] & \\ & \mathfrak{p}\Lambda, \mathfrak{p}\Pi & \end{tikzcd}$$

<u>EX</u>: $k=1$, $F=\mathbb{Q}$, $Q(\Lambda)\mathbb{Z}=\mathbb{Z}$

$$\{\Pi \sim_p \Lambda\} \xleftrightarrow{\sim} Q_\Lambda(\mathbb{F}_p)$$

$$Q_\Lambda \subseteq \mathbb{P}^{m-1}_{\mathbb{F}_p}$$

$$p^{-1} v\mathbb{Z} + \{w \in V : B(v,w) \in p\mathbb{Z}\} \longmapsto v$$

<u>p</u> GOOD

$$\Lambda \otimes R_{\mathfrak{p}} \simeq H \boxplus \dots \boxplus H \boxplus \Lambda'_{\mathfrak{p}}$$

ANISOTROPIC ($Q(x)=0 \Rightarrow x=0$).

$H \cong R_{\mathfrak{p}}^2$ HYPERBOLIC PLANE.

$$Q|_H(x,y) = xy \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$\mathfrak{p} = \pi R_{\mathfrak{p}}$

$$\begin{pmatrix} \pi & 0 \\ 0 & \pi^{-1} \end{pmatrix} \in O(H)(F_{\mathfrak{p}})$$

COMPLETION

$$T_{\mathfrak{p},k} = T_{\hat{\pi}_k}, \quad \hat{\pi}_k = \begin{pmatrix} \overbrace{\pi\ \pi^{-1}\ \ddots\ \pi\ \pi^{-1}}^{k} & \\ & 1 \ \ddots \ 1 \end{pmatrix} \in \hat{G}.$$

$$\hat{K}\hat{\pi}_k\hat{K} = \bigsqcup_j \hat{\pi}_{kj}\hat{K}$$

$$\Pi_{kj} := \hat{\pi}_{kj}\hat{\Lambda} \cap V$$ j^{k} NEIGHBOR.

$$T_{p,k} : M_W(\hat{K}) \to M_W(\hat{K}),$$

$$(T_{p,k} f)([\Lambda_i]) = \sum_{\Pi \sim_{p,k} \Lambda_i} f([\Pi]).$$ W trivial

MORE GENERALLY,
IF $\Pi = g\Lambda'$, $f([\Pi]) = g \cdot f([\Lambda'])$.

<u>EX</u>: W trivial. $[T_p] = (m_{ij})_{i,j}$.

$$m_{ij} = \#\{\Pi \sim_p \Lambda_i : \Pi \cong \Lambda_j\}.$$

ADJACENCY MATRIX OF p-NEIGHBOR GRAPH

<u>EX</u>: $\det = 11^2$

$$[T_2] = \begin{pmatrix} 1 & 9 & 3 \\ 4 & 0 & 0 \\ 4 & 0 & 6 \end{pmatrix}, \quad [T_3] = \begin{pmatrix} 1 & 4 & 2 \\ 9 & 4 & 6 \\ 6 & 8 & 8 \end{pmatrix}$$

<u>THM</u> (HEIN): FIX $m \geq 1$. THEN THE PROBLEM

GIVEN: $\Lambda, \mathfrak{p}, k$

RETURN: $[T_{\mathfrak{p},k}] \circlearrowright M(\hat{K})$ trivial weight.

$h := \dim M(\hat{K})$

CAN BE SOLVED IN TIME

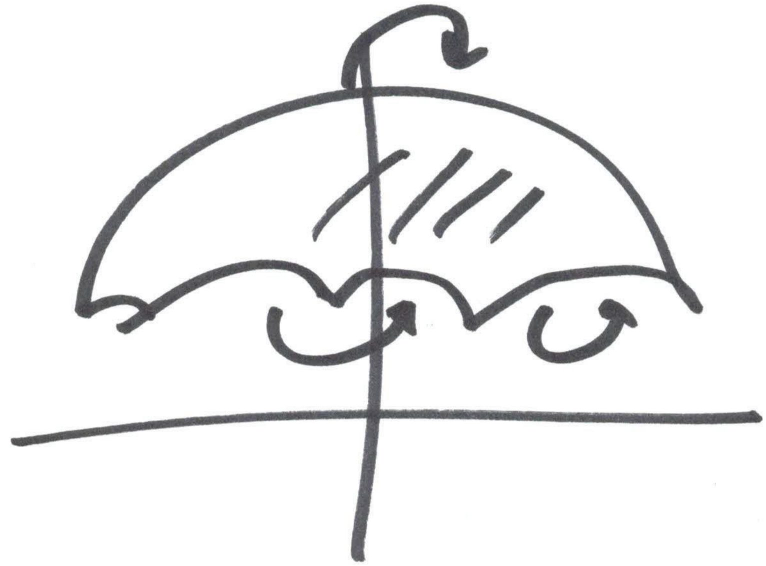
$$(N m \mathfrak{p})^{k(m-k-1)} \cdot h^2.$$ POLYNOMIAL IN INPUT SIZE.

<u>RMK</u>: IN HIGHER WEIGHT, MUST INCLUDE RUNNING TIME FOR COMPUTING $G \circlearrowright W$.

GLIMPSE OF OTHER APPROACHES.

1. ALGEBRAIC MODULAR FORMS
2. DIRECT GROUP COHOMOLOGICAL METHOD.

EX: SHIMURA CURVES. H^1



IDEA: COMPUTE A FUNDAMENTAL DOMAIN FOR $\Gamma \circlearrowright D_\infty$, GIVING A FINITE, COMPUTABLE PRESENTATION FOR Γ.

(3) GENERALIZED MODULAR SYMBOLS.

<u>IDEA</u>: REPLACE CUSPS BY PARABOLIC SUBGROUPS

BOUNDARY IN ~~BAILLY-BOREL~~ BOREL-SERRE COMPACTIFICATIONS

<u>EX</u>: $SL_3(\mathbb{Z})$.

(4) SHARBLY COMPLEX:

<u>IDEA</u>: DEFORMATION RETRACT

(5) EXPLICIT LIFTS OR CONSTRUCTIONS

(6) TRACE FORMULA

(7) MODULI METHODS