COMPUTING MODULAR FORMS

(DAY 3)

LAST TIME:

G (SEMISIMPLE OR) REDUCTIVE LINEAR ALGEBRAIC GROUP OVER A NUMBER FIELD F.

$F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^r \times \mathbb{C}^s$

$G_\infty := G(F_\infty)$

$K_\infty \leq G_\infty$ MAXIMAL COMPACT

$A_\infty = S_G(F_\infty)^\circ$ $\{= 1$ IF G SEMISIMPLE$\}$

$D_\infty := G_\infty / A_\infty K_\infty$

EX: $G = GL_2, SL_2$ OVER $F = \mathbb{Q}$.

$D_\infty = \mathfrak{H}^2$

GOAL: $Y(\hat{K}) = \bigsqcup_i \Gamma_i \backslash D_\infty$

COMPUTE $H^*(Y(\hat{K}), W)$.

LET $R := \mathbb{Z}_F$ BE THE RING OF INTEGERS OF F.

FOR $\mathfrak{a} \subset F$ A FRACTIONAL R-IDEAL, $\mathfrak{p}$ PRIME AND $R_{\mathfrak{p}}$ THE COMPLETION,

$$\mathfrak{a} \otimes R_{\mathfrak{p}} = \alpha_{\mathfrak{p}} R_{\mathfrak{p}} \in F_{\mathfrak{p}}^{\times} / R_{\mathfrak{p}}^{\times},$$

$\alpha_{\mathfrak{p}} \in R_{\mathfrak{p}}^{\times}$ FOR ALL BUT FINITELY MANY $\mathfrak{p}$.

LET
$$\hat{F}^{\times} := \prod_{\mathfrak{p}}{}' F_{\mathfrak{p}}^{\times}$$
$$= \{ (\alpha_{\mathfrak{p}})_{\mathfrak{p}} : \alpha_{\mathfrak{p}} \in R_{\mathfrak{p}}^{\times} \text{ A.A. } \mathfrak{p} \}$$
$$\cup\!\!|$$
$$\hat{R}^{\times} := \prod_{\mathfrak{p}} R_{\mathfrak{p}}^{\times}$$

FINITE IDELES.

IDELES: $\underset{\sim}{F}^{\times} := \hat{F}^{\times} \times F_{\infty}^{\times}$

GROUP OF FRACTIONAL IDEALS IS CANONICALLY ISOMORPHIC TO $\hat{F}^\times / \hat{R}^\times$.

$\Rightarrow \mathrm{Cl}\, R \cong F^\times \backslash \hat{F}^\times / \hat{R}^\times$.

(GL_1)

EX: $F = \mathbb{Q}$, $R = \mathbb{Z}$,

$\mathbb{Q}^\times \backslash \hat{\mathbb{Q}}^\times / \hat{\mathbb{Z}}^\times = \{\cdot\}$.

SIMILARLY,

$$\hat{F} := \prod_{\mathfrak{p}}{}' F_{\mathfrak{p}}$$
$$\vee\!\!|$$
$$\hat{R} := \prod_{\mathfrak{p}} R_{\mathfrak{p}}.$$

$$\underline{F} := \hat{F} \times F_\infty$$

LET $\hat{G} := \mathbf{G}(\hat{F})$.

LET $\hat{K} \leq \hat{G}$ COMPACT OPEN SUBGROUP, <u>LEVEL</u>.

CHOOSE MODEL $\mathcal{G}$ OVER R

$$\pi_N : \mathcal{G}(\hat{R}) \longrightarrow \mathcal{G}(R/N)$$
$$\hat{K} \longmapsto K_N .$$

$N \subseteq R$ NONZERO. IDEAL.

LET $G := \mathbf{G}(F)$.

$$Y := Y(G, \hat{K})$$
$$= G \backslash (D_\infty \times \hat{G}/\hat{K})$$
$$= G \backslash (G_\infty / A_\infty K_\infty \times \hat{G}/\hat{K})$$
$$= G \backslash \underline{G} / \underline{K}$$

<u>LOCALLY SYMMETRIC SPACE</u>.

THM: $G \backslash \hat{G} / \hat{K}$ IS A FINITE SET.

THINK: CLASS SET.

"PROOF": GEOMETRY OF NUMBERS

WRITE $G \backslash \hat{G} / \hat{K} = \bigsqcup_i G \hat{\beta}_i \hat{K}$,

$\underbrace{\Gamma_i}_{\leq G} := \hat{\beta}_i \hat{K} \hat{\beta}_i^{-1} \cap G$ ARITHMETIC GROUP

THEN $Y(\hat{K}) = \bigsqcup_i \Gamma_i \backslash D_\infty$

IS A FINITE DISJOINT UNION OF REAL ORBIFOLDS

$Y \subseteq X = X(G, \hat{K})$ ~~BAILY~~-BOREL-SERRE COMPACTIFICATION.

LET $\rho: G \to GL(W_\rho) \simeq GL_{m,E}$

BE A <u>LINEAR REPRESENTATION</u>

OF G OVER E, char $E=0$

CALL $W=W_\rho$ THE <u>WEIGHT</u>.

DEFINES A LOCALLY CONSTANT SHEAF $\mathcal{W}$ ON $Y(\hat{K})$.

<u>DEF</u>: <u>SPACE OF (COHOMOLOGICAL) MODULAR FORMS</u> FOR G OF LEVEL $\hat{K}$, WEIGHT W, AND DEGREE r, IS

$$M_{W,r}(\hat{K}) := H^r(Y(\hat{K}), \mathcal{W}) \simeq \bigoplus_i H^r(Y(\Gamma_i), \mathcal{W})$$

EACH $Y(\Gamma_i) \leq \Gamma_i \backslash D_\infty$ HAS D_∞ CONTRACTIBLE, $H^r(Y(\Gamma_i), \mathcal{W}) \simeq H^r(\Gamma_i, W)$

HECKE OPERATORS:

FOR EACH $\hat{K}\hat{\pi}\hat{K} \in \hat{K}\backslash\hat{G}/\hat{K}$

THINK: CHOICE OF INVARIANT FACTORS.

WRITE: $\hat{K}\hat{\pi}\hat{K} = \bigcup_j \hat{\pi}_j \hat{K}$

$\hat{K}' := \hat{K} \cap \hat{\pi}\hat{K}\hat{\pi}^{-1}$.

$$Y(\hat{K}') = G\backslash(D_\infty \times \hat{G}/\hat{K}')$$

$$\begin{tikzcd}
 & (z,\hat{g}\hat{K}') \in Y(\hat{K}') \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & (z,\hat{g}\hat{K}') \arrow[d, mapsto] \\
(z,\hat{g}\hat{K}) \in Y(\hat{K}) & & Y(\hat{K}) \quad (z,\hat{g}\hat{\pi}\hat{K}).
\end{tikzcd}$$

$T_{\hat{\pi}} = (s_2)_* \circ s_1^*$ INDUCES ACTION ON $M_{w,r}(\hat{K})$.

$r=0$: $(T_{\hat{\pi}} f)(z,\hat{g}\hat{K}) = \sum_j f(z,\hat{g}\hat{\pi}_j\hat{K})$.

GIVEN: G OVER F

LEVEL $\hat{K} \subseteq \hat{G}$, $\hat{K} = \pi_N^{-1}(K_N)$

WEIGHT w

DEGREE $r \geq 0$

OUTPUT: ① $\dim M_{G,w,r}(\hat{K})$

② FOR $\hat{\pi} \in \hat{G}$, $[T_{\hat{\pi}}] \circlearrowright M_{G,w,r}(\hat{K})$.

③ SERIES EXPANSIONS,
L-FUNCTIONS,
GALOIS REIRESENTATIONS,
...

<u>EX</u>: $G = GL_{2,\mathbb{Q}}$, $\hat{\pi} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

$Y(\hat{K}) = G \backslash (D_\infty \times \hat{G}/\hat{K})$

$D_\infty = \{\cdot\}$?

<u>DEF</u>: G IS <u>TOTALLY DEFINITE</u>

(<u>COMPACT AT ∞ MOD CENTER</u>)

IF G_∞ / A_∞ IS COMPACT

$\Leftrightarrow G_\infty / A_\infty K_\infty = D_\infty = \{\cdot\}$.

<u>PROP</u>: IF G IS ALMOST SIMPLE AND TOTALLY DEFINITE, THEN:

(a) F IS TOTALLY REAL, AND

(b) FOR ALL (REAL) $v | \infty$, UP TO CENTRAL ISOGENY, G_v IS ONE OF

$SU(n+1), SO(2n+1), Sp(n), SO(2n),$

OR EXCEPTIONAL

(I.E. PRESERVES DEFINITE QUADRATIC,

HERMITIAN, OR QUATERNION HERMITIAN FORM; OR EXCEPTIONAL)

$$\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$$

$$Sp(n) = USp(2n) = Sp(2n,\mathbb{C}) \cap U(2n)$$
$$= U(n,\mathbb{H}).$$

SUPPOSE G IS TOTALLY DEFINITE.

$$Y(\hat{K}) = G \backslash \cancel{D_\infty} \times \hat{G}/\hat{K} = G \backslash \hat{G}/\hat{K}$$

FINITE SET.

<u>LEM</u>:
EACH Γ_i IS FINITE.

<u>PROOF</u>: EACH $\Gamma_i \leq G_\infty/A_\infty$ IS DISCRETE IN A COMPACT (HAUSDORFF) GROUP, FINITE.

$r=0$

$$M_W(\hat{K}) = H^0(Y(\hat{K}), W)$$
$$= \oplus H^0(\Gamma_i, W)$$

<u>EX</u>: $G = GL_{1,F}$, $R = \mathbb{Z}_F$.

$G \backslash \hat{G} / \hat{K} = Y = Cl\, R$.

$G_\infty = F_\infty^\times$

$K_\infty = \{\pm 1\}^r \times (\mathbb{C}^1)^c$

$A_\infty = (\mathbb{R}^\times_{>0})^r \times (\mathbb{C}^\times)^c$ ← UNIT CIRCLE.

$W \simeq \mathbb{C}$ INFINITY TYPE.

$f \in M_W(\hat{K})$ EIGENFORM

$\Leftrightarrow f = \psi : \hat{F}^\times \longrightarrow \mathbb{C}^\times$

HECKE QUASI-CHARACTERS.

$F = \mathbb{Q}(i)$, $\hat{K} = 1 + (2+2i)\widehat{\mathbb{Z}[i]}$

GAUSS BIQUADRATIC.

<u>EX</u>: $G_\infty = GSp(1) \simeq \mathbb{H}^\times$

$F = \mathbb{Q}$,

$G = B^\times$,

B DEFINITE QUATALG OVER $\mathbb{Q}$. $(B \otimes \mathbb{R} \simeq \mathbb{H})$.

$\mathbb{H} \circlearrowleft \mathbb{H}$ RIGHT $\mathbb{H}$-VECTOR SPACE

$G \circlearrowright \mathbb{H}$

$\mathbb{H} = \mathrm{End}_{\mathbb{H}}(\mathbb{H})$

$\rightsquigarrow G$

$\hat{K} = \hat{\mathcal{O}}^\times$, $\mathcal{O} \subseteq B$ (MAXIMAL) ORDER

$\hat{B}^\times / \hat{\mathcal{O}}^\times = \hat{G}/\hat{K}$

$= \left\{ \begin{array}{c} \text{LOCALLY PRINCIPAL} \\ \text{RIGHT } \mathcal{O}\text{-IDEALS} \end{array} \right\}$

$Y = B^\times \backslash \hat{B}^\times / \hat{\mathcal{O}}^\times = \mathrm{Cls}\, \mathcal{O}$.

$[I] = [J] \Leftrightarrow I = \alpha J,\ \alpha \in B^\times$.

$\mathrm{disc}\, B = p$: BRANDT MATRIX

$T_\pi = T_\ell$: ℓ-ISOGENY MATRIX.