COMPUTING MODULAR FORMS

(DAY 2)

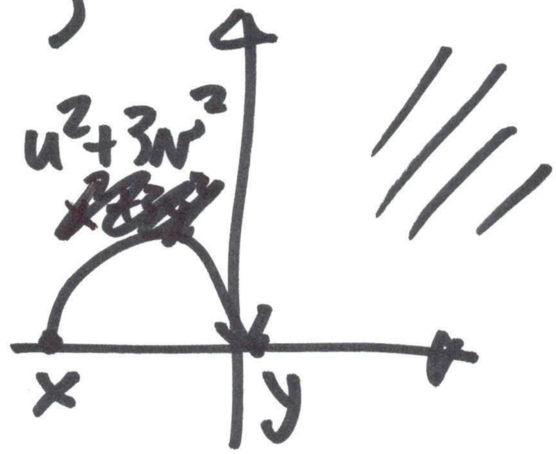
GOAL: GENERALIZATIONS.

LAST TIME: COMPUTED A HECKE ACTION ON $S_k(\Gamma)$ VIA MODULAR SYMBOLS (VIA DUALITY)

$$H_1(X(\Gamma), \mathrm{Sym}^{k-2}\mathbb{C}^2)$$

$\{T_p\}_p$.

$(u^2+3v^2) \otimes \{x,y\}$.



$$\Gamma \leq SL_2(\mathbb{Z}) \circlearrowright \mathcal{H}^2 \rightsquigarrow Y(\Gamma) = \Gamma \backslash \mathcal{H}^2$$

$SL_2(\mathbb{Z}) \subseteq SL_2(\mathbb{Q})$

$Y(\Gamma) \subseteq X(\Gamma)$.

OVERVIEW:

$$\Gamma_i \leq G(F) \circlearrowright D_\infty$$

ARITHMETIC GROUP

REDUCTIVE ALGEBRAIC GROUP OVER NUMBER FIELD $/F$

GLOBAL SYMMETRIC SPACE

$Y(\Gamma_i) := \Gamma_i \backslash D_\infty$

$$\hat{K} \leq G(\hat{F}) \rightsquigarrow Y(\hat{K}) = \coprod_i^{<\infty} Y(\Gamma_i) \subseteq X(\hat{K}).$$

LEVEL

$G \circlearrowright W$ WEIGHT

COMPUTE $\{T_{\hat{\pi}}\}_{\hat{\pi}} \circlearrowright H^r(Y(\hat{K}), W)$

$$\cong \bigoplus_i H^r(Y(\Gamma_i), W)$$

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r DEGREE

LET F BE A FIELD, $\mathrm{char}\, F = 0$, F^{al} ALGEBRAIC CLOSURE.

$$\underset{\substack{\| \\ Gl_n}}{Gl_{n,F}} := \mathrm{Spec}\, F[x_{ij}]_{i,j=1,\dots,n}[D^{-1}]$$

$$D = \det(x_{ij})_{i,j}.$$

GROUP OBJECT IN CATEGORY OF AFFINE SCHEMES OF FINITE TYPE OVER F.

E.G. $m : Gl_n \times Gl_n \longrightarrow Gl_n$

$$((x_{ij}),(y_{ij})) \longmapsto (z_{ij})$$

$$z_{ij} = \sum_{r=1}^{n} x_{ir} y_{rj}.$$

$$m^* : F[Gl_n] \longrightarrow F[Gl_n] \otimes_F F[Gl_n]$$

$$m^*((x_{ij})) = \sum_{r=1}^{n} x_{ir} \otimes x_{rj}.$$

YONEDA'S LEMMA.

<u>DEF</u>: A <u>LINEAR ALGEBRAIC GROUP</u> G OVER F IS A CLOSED SUBGROUP SCHEME OF $GL_{n,F}$.

<u>EX</u>: $GL_1 = \mathbb{G}_m$

$\mathbb{G}_a \leq GL_2$

$$\mathbb{G}_a = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} : \begin{array}{l} x_{11} = x_{22} = 1 \\ x_{21} = 0 \end{array}$$

LET G BE A LINEAR ALGEBRAIC GROUP. LET G^0 BE THE CONNECTED COMPONENT OF IDENTITY. THEN $\pi_0(G) := G/G^0$ IS FINITE, AND OVER $F = F^{al}$

$$G = \bigsqcup_{g \in \pi_0(G)(F^{al})} g G^0.$$

<u>EX</u>: $Q(x_1, \dots, x_n) \in F[x_1, \dots, x_n]_2$

QUADRATIC FORM OVER F.

HOMOGENEOUS OF deg 2

E.G. $Q = x_1^2 + \dots + x_n^2$. NONDEGENERATE.

ASSOCIATED BILINEAR FORM $T \in M_n(F)$. ORTHOGONAL GROUP

$$O_n(Q)(R) := \{ A \in GL_n(R) : A^t T A = T \}$$

R F.G. F-ALGEBRA.

IS A LINEAR ALGEBRAIC GROUP.

$O_n(Q) = SO_n(Q) \sqcup g\, SO_n(Q)$,

$\det(g) = -1$, $g \in O_n(Q)(F)$.

$SO_n(Q) \leq O_n(Q)$, det 1.

<u>DEF</u>: G IS <u>SEMISIMPLE</u> IF THE MAXIMAL CONNECTED SOLVABLE NORMAL SUBGROUP IS TRIVIAL.

<u>SLOGAN</u>: NO $\mathbb{G}_m$'S, NO $\mathbb{G}_a$'S.

<u>DEF</u>: G IS <u>ALMOST SIMPLE</u> IF G IS SEMISIMPLE, NONTRIVIAL, AND EVERY CONNECTED NORMAL SUBGROUP IS TRIVIAL OR G.

CLASSIFICATION (BY ROOT DATUM): OVER F^a AND UP TO CENTRAL ISOGENY, THE ALMOST SIMPLE LINEAR ALGEBRAIC GROUPS ARE

...

CLASSICAL	A_{n-1}	SL_n
	B_n	SO_{2n+1}
	C_n	Sp_{2n}
	D_n	SO_{2n}

EXCEPTIONAL E_6, E_7, E_8, F_4, G_2.

<u>DEF</u>: A TORUS OVER F IS AN ALGEBRAIC GROUP T S.T. $T^{al} \simeq \mathbb{G}_m^r$. <u>SPLIT</u> IF $T \simeq \mathbb{G}_m^r$ (OVER F).

$" T \times_F F^{al}$

<u>RANK</u> OF G IS THE DIMENSION OF A MAXIMAL SPLIT TORUS IN G.

$S_G :=$ MAXIMAL SPLIT TORUS IN CENTER $Z(G)$.

S_G = TRIVIAL IF G IS SEMISIMPLE.

$$1 \to SL_n \to GL_n \xrightarrow{\det} \mathbb{G}_m \to 1.$$

SEMISIMPLE REDUCTIVE

<u>DEF</u>: G IS <u>REDUCTIVE</u> IF THE MAXIMAL CONNECTED UNIPOTENT NORMAL SUBGROUP IS TRIVIAL.

<u>PROP</u>: IF G IS REDUCTIVE, THEN THERE IS AN EXACT SEQUENCE

$$1 \to G^{der} \to G \to G/G^{der} \to 1.$$

DERIVED GROUP SEMISIMPLE

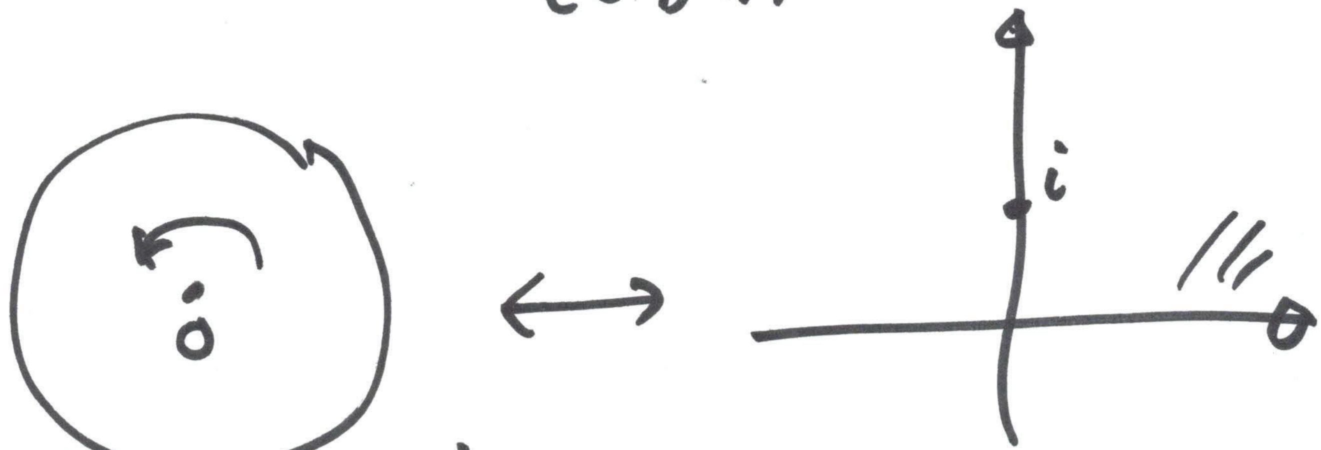
TORUS

SURE, $SL_2(\mathbb{R}) \circlearrowright \mathcal{H}^2$.

$SL_2(\mathbb{R}) \circlearrowright \underset{\displaystyle G_\infty}{\underset{\parallel}{SL_2(\mathbb{R})}}$ WITH

$\mathrm{stab}_{SL_2(\mathbb{R})}(i) = SO(2) = K_\infty$.

$= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in SL_2(\mathbb{R}) \right\}$.



$w = \dfrac{z-i}{z+i} \longleftarrow\!\shortmid\ z$

HOMEOMORPHISM: $G_\infty / K_\infty \xrightarrow{\sim} \mathcal{H}^2$

$gK_\infty \longmapsto gi$

PRESERVES METRIC: $\underbrace{\|g\|^2}_{a^2+b^2+c^2+d^2} = 2\cosh \rho(i, gi)$.

$Y(\Gamma) = \Gamma \backslash \mathcal{H}^2 = \Gamma \backslash G_\infty / K_\infty$.

DOUBLE COSETIFICATION

$F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^r \times \mathbb{C}^c$

FOR $\underline{v|\infty}$, $G_v := G(F_v)$

$$G_\infty := G(F_\infty) \simeq \prod_{v|\infty} G(F_v).$$

EACH G_v HAS A MAXIMAL COMPACT SUBGROUP $K_v \leq G_v$, UNIQUE UP TO CONJUGATION.

$$K_\infty := \prod_{v|\infty} K_v \leq G_\infty.$$

$A_\infty := S_G(F_\infty)^\circ$ ← CONNECTED COMPONENT IN REAL TOPOLOGY.

$D_\infty := G_\infty / A_\infty K_\infty$ <u>GLOBAL SYMMETRIC DOMAIN</u>

<u>PROP</u>: D_∞ IS CONTRACTIBLE, DIFFEOMORPHIC TO EUCLIDEAN SPACE, HAS A TRANSITIVE ACTION BY G_∞.

<u>EX</u>: $G = GL_{2,F}$

$G_\infty = GL_2(\mathbb{R})^r \times GL_2(\mathbb{C})^c$

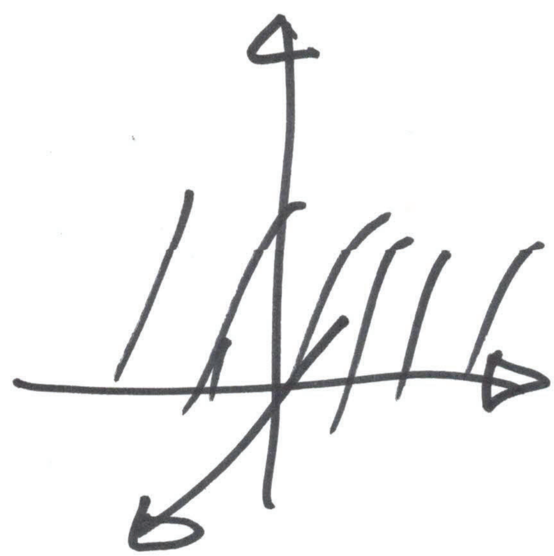
$K_\infty = O(2)^r \times U(2)^c$

$A_\infty = (\mathbb{R}^\times_{>0})^r \times (\mathbb{C}^\times)^c$

$D_\infty = (\mathcal{H}^2)^r \times (\mathcal{H}^3)^c$

$$\mathcal{H}^3 = \{(x,y,t) \in \mathbb{R}^3 : t > 0\}$$

$\dim_{\mathbb{R}} D_\infty = 2r + 3c.$



<u>EX</u>: $G = GL_{n,\mathbb{Q}}$.

$G_\infty = GL_n(\mathbb{R})$

$K_\infty = O(n)$

$A_\infty = \mathbb{R}^\times_{>0}$

$D_\infty = GL_n(\mathbb{R}) / \mathbb{R}^\times_{>0} O(n)$.

$$\dim D_\infty = \frac{n(n+1)}{2} - 1.$$

$D_\infty = \{\text{SHAPES OF FRAMED LATTICES}\}$

$$= \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ x_{ij} & & t_n \end{pmatrix} : \begin{array}{l} t_i > 0, \\ x_{ij} \in \mathbb{R} \end{array} \right\} \Big/ \mathbb{R}^\times_{>0}$$

SCALE SHEAR.

$n = 2$: