COMPUTING MODULAR FORMS (DAY 1)

TODAY: MODULAR SYMBOLS

(BIRCH, CREMONA, MANIN, MEREL, STEIN, ASHF)

$$SL_2(\mathbb{R}) \circlearrowright \mathcal{H}^2 := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

$$\circlearrowright \mathcal{H}^{2*} := \mathcal{H}^2 \cup \overbrace{\mathbb{P}^1(\mathbb{Q})}^{\text{CUSPS}}$$

$$\begin{array}{c} \vee\!\mid \\ SL_2(\mathbb{Z}) \end{array} \text{ DISCRETE}$$

$$SL_2(\mathbb{Z}) \backslash \mathcal{H}^2 \leftrightarrow \left\{ \begin{array}{c} \text{ORIENTED LATTICES} \\ \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subseteq \mathbb{C} \\ \mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0. \end{array} \right\} \Big/ \underset{\text{HOMOTHETY}}{\sim}$$

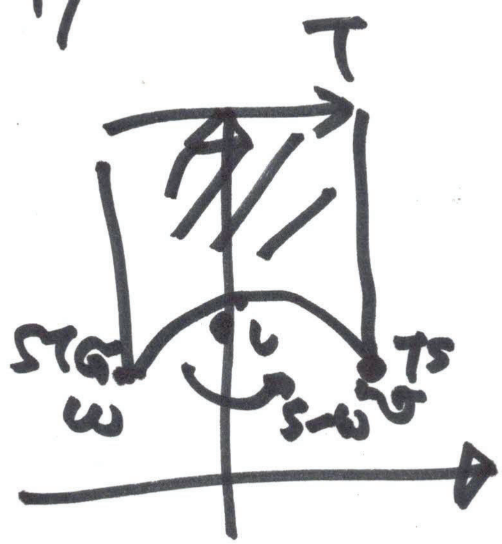
$$SL_2(\mathbb{Z})z \mapsto [\mathbb{Z}z + \mathbb{Z}].$$

$$SL_2(\mathbb{Z}) = \langle S, T \mid S^2 = (ST)^3 = -1 \rangle$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$z \mapsto -\frac{1}{z} \qquad z \mapsto z+1.$$

$$ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$



REDUCTION ALGORITHM
ALSO SOLVES WORD PROBLEM: ON $\mathbb{P}^1(\mathbb{Q})$
MANIFESTS AS EUCLIDEAN ALGORITHM.

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \quad \gamma(\infty) = \frac{a}{c}, \ c \neq 0.$$

$$a = qc + r \quad \begin{cases} 0 \leq r < |c| \\ -\frac{1}{2} < \frac{r}{c} \leq \frac{1}{2} \\ |c| < r \leq 0 \end{cases}$$

$$T^{-q}\gamma = \begin{pmatrix} r & b-qd \\ c & d \end{pmatrix}, \quad ST^{-q}\gamma = \begin{pmatrix} -c & -d \\ r & b-qd \end{pmatrix}, \ \ldots$$

$$\frac{a}{c} = q + \frac{r}{c} = q - \frac{1}{\left(\frac{-c}{r}\right)} = q_1 - \cfrac{1}{q_2 - \cdots \cfrac{1}{q_r}}$$

$$\gamma = \pm T^{a_1} S T^{a_2} \cdots S T^{a_r} S T^{b}$$

CONVERGENTS $\frac{a_1}{c_1} = a_1, \frac{a_2}{c_2}, \ldots, \frac{a_r}{c_r} = \frac{a}{c}$

$$\begin{pmatrix} a_i & a_{i+1} \\ c_i & c_{i+1} \end{pmatrix} \in \pm SL_2(\mathbb{Z}).$$

Ex: $\frac{17}{5} = 3 + \frac{2}{5} = 3 - \frac{1}{\left(\frac{-5}{2}\right)} = 3 - \frac{1}{-2 - \frac{1}{2}}$

$$\frac{3}{1}, \frac{7}{2}, \frac{17}{5}$$

LET $\Gamma \leq SL_2(\mathbb{Z})$ BE A CONGRUENCE SUBGROUP, DEFINED BY $\pi^{-1}(K_N)$.

$$\begin{array}{l} \pi : SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z}) \hookrightarrow GL_2(\mathbb{Z}/N\mathbb{Z}) \\ \qquad\quad \Gamma \longmapsto \qquad\qquad\qquad\qquad\quad\; K_N \leq GL_2(\mathbb{Z}/N\mathbb{Z}) \end{array}$$

IF N IS MINIMAL, SAY <u>LEVEL</u>.

CosetEnum:

GIVEN: K_N

RETURN: $\{\bar{g}_i\}_i$ S.T. $K_N \backslash GL_2(\mathbb{Z}/N\mathbb{Z}) = \bigsqcup_i K_N \bar{g}_i$

CosetIndex:

GIVEN: K_N, $\{\bar{g}_i\}_i$, $\bar{g} \in GL_2(\mathbb{Z}/N\mathbb{Z})$

RETURN: i S.T. $\bar{g} \in K_N \bar{g}_i$.

FAREY SYMBOLS.

EX: FOR $\Gamma = \Gamma_0(N) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N$

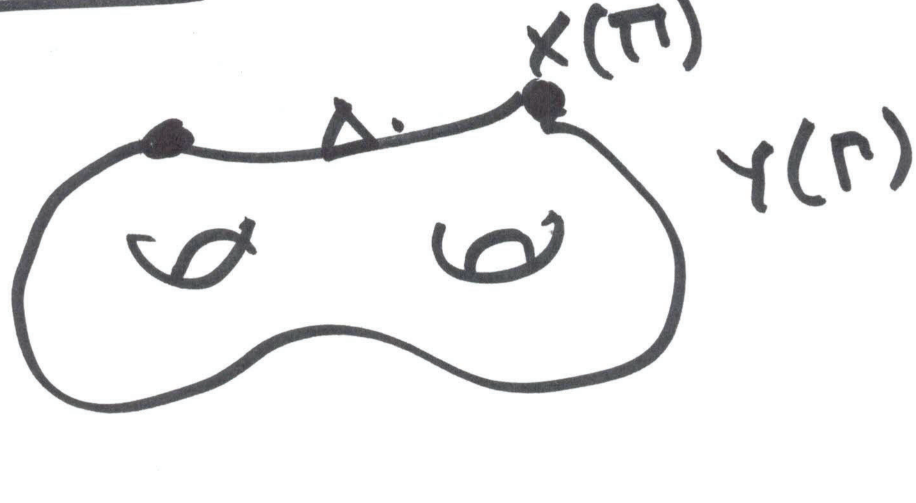
$\Gamma_0(N) \longleftrightarrow K_N$

$i = (c:d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, $\bar{g}_i = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$.

SUPPOSE $\det(K_N) = (\mathbb{Z}/N\mathbb{Z})^\times$

LET $Y(\Gamma) := \Gamma \backslash \mathcal{H}^2 \subseteq X(\Gamma) := \Gamma \backslash \mathcal{H}^{2*}$

CUSPS $X(\Gamma) \setminus Y(\Gamma) = \Delta$



DEF: A <u>MODULAR FORM</u> $f \in M_k(\Gamma)$ OF <u>WEIGHT</u> $k \in \mathbb{Z}_{\geq 2}$ AND <u>LEVEL</u> Γ IS A HOLOMORPHIC MAP $f : \mathcal{H}^2 \longrightarrow \mathbb{C}$ S.T.

$$(*) \quad f(\gamma z) = (cz+d)^k f(z) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad \forall z \in \mathcal{H}^2$$

AND f EXTENDS TO A HOLOMORPHIC FUNCTION ON $\mathcal{H}^{2*}$; <u>CUSP FORM</u> IF f VANISHES AT CUSPS.
$\in S_k(\Gamma)$

FOR $\alpha \in GL_2(\mathbb{R})_{>0}$ ← POSITIVE DETERMINANT

WRITE $f[\alpha]_k(z) = (\det \alpha)^{k-1}(cz+d)^{-k} f(\alpha z)$

$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$(*) \Leftrightarrow f[\gamma]_k(z) = f(z).$$

HECKE OPERATORS AS AVERAGING OPERATORS.

$$\mathbb{Z}^2 \simeq \Lambda \subseteq \Lambda_0 \simeq \mathbb{Z}^2 \qquad [\Lambda_0 : \Lambda] = n$$

GIVES $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) M_2(\mathbb{Z})_n GL_2(\mathbb{Z})$

$$= \bigsqcup_{\substack{d \mid a \\ ad = n}} GL_2(\mathbb{Z}) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} GL_2(\mathbb{Z})$$

<u>INVARIANT FACTORS</u>

$\Lambda_0 / \Lambda \simeq \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ (SMITH NORMAL FORM).

<u>PRIMITIVE</u> IF $d=1$.

EXTENDS TO ORIENTED WITH $SL_2(\mathbb{Z})$.

$$\left\{\begin{array}{c}\Lambda \subseteq \Lambda_0 \text{ PRIMITIVE} \\ \text{WITH FIXED} \\ \text{INVARIANT FACTORS} \\ (n,1)\end{array}\right\} \qquad GL_2(\mathbb{Z})\begin{pmatrix} n & 0 \\ 0 & 1\end{pmatrix}GL_2(\mathbb{Z}) = \bigsqcup_{\alpha \in \Theta_n} \alpha\, GL_2(\mathbb{Z})$$

$$\Big\updownarrow 2$$

$$\Theta_n := \left\{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} : \begin{array}{c} ad = n \\ \gcd(a,d)=1 \\ 0 \le b < d\end{array}\right\} \ni \alpha.$$

<u>EX</u>: $n = p$, $\Theta_p = \left\{\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & p\end{pmatrix}, \ldots, \begin{pmatrix} 1 & p-1 \\ 0 & p\end{pmatrix}\right\}.$

$$\#\Theta_n = \#\mathbb{P}^1(\mathbb{Z}/n\mathbb{Z}).$$

LET $\beta \in GL_2(\mathbb{Q})_{>0}$

LEM: $\Gamma\beta\Gamma = \bigsqcup^{<\infty}_{\alpha \in \Theta(\beta)} \alpha\Gamma$

WITH $\# \Theta(\beta) < \infty$.

DEF: **HECKE OPERATOR**

$$T_\beta : M_k(\Gamma) \longrightarrow M_k(\Gamma)$$

$$T_\beta(f) = \sum_{\alpha \in \Theta(\beta)} f[\alpha]_k$$

EX: FOR $p \nmid N$, $T_p = T_\beta$, WHERE $\beta = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

FOR $p \nmid N$, THESE T_p ARE ~~SELF-ADJOINT~~ NORMAL, PAIRWISE COMMUTE
$\Rightarrow$ BASIS OF EIGENFORMS.

THM (ASSAF):

GIVEN: K_N, $k \in \mathbb{Z}_{\geq 2}$, $p \nmid N$

RETURN:
$$\underset{M_h(\mathbb{Q})}{\underset{\in}{[T_p]}} \circlearrowright \underset{S_k(\Gamma).}{\underset{\cup|}{M_k(\Gamma)}}$$

COMPUTED IN TIME

$$O(\underbrace{CE}_{\text{TIME FOR CosetEnum}} + \underbrace{CI}_{\text{TIME FOR CosetIndex.}} \cdot p \log p \cdot h)$$

$$h := \dim S_k(\Gamma) \approx k[SL_2(\mathbb{Z}) : \Gamma]$$

$$\frac{d(\gamma z)}{dz} = (cz+d)^{-2}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$\Rightarrow f \in M_2(\Gamma),\ \gamma \in \Gamma$, $\quad f(\gamma z)\, d(\gamma z) = f(z)\, dz.$

$dz = \frac{1}{2\pi i} \frac{dq}{q}$ SO $M_2(\Gamma) \cong \Omega^1(\Delta)$

ALLOW SIMPLE POLES AT CUSPS

$q = e^{2\pi i z}$

$$\begin{array}{c} \cup \! \mid \\ S_2(\Gamma) \cong \Omega^1. \end{array}$$

RMK: FOR $k \in \mathbb{Z}_{\geq 2}$, WORK WITH

$$\omega_f(z) := f(z) \begin{pmatrix} z \\ 1 \end{pmatrix}^{\otimes (k-2)} dz$$

$\begin{pmatrix} z \\ 1 \end{pmatrix}^{\otimes k-2} \in \mathrm{Sym}^{k-2}\, \mathbb{C}^2 \circlearrowleft GL_2(\mathbb{Q}).$

THM: THE INTEGRATION PAIRING

$$\underset{\underset{\underset{\mathbb{C}^g}{\wr\wr}}{S_2(\Gamma)}}{\overset{\wr\parallel}{\Omega^1(X(\Gamma))}} \times H_1(X(\Gamma), \cancel{\mathbb{Z}})_{\mathbb{R}} \longrightarrow \mathbb{C}$$

$$H_1(X(\Gamma), \mathbb{Z}) \cong \cancel{\mathbb{Z}}^{2g}\ \mathbb{R}$$

$$\langle \omega, \gamma \rangle = 2\pi i \int_\gamma \omega, \qquad \omega = f(z)\,dz$$

$$= 2\pi i \int_\gamma f(z)\, dz$$

IS PERFECT AS $\mathbb{R}$-VECTOR SPACES.

COR: $H_1(Y(\Gamma), \mathbb{C})^\vee$ ← DUAL

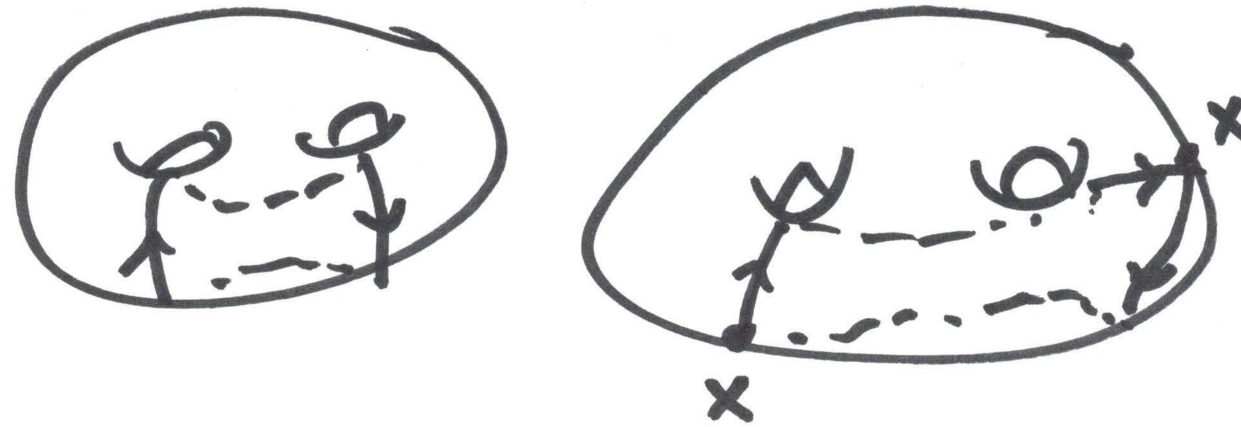
$$\simeq M_{\cancel{k}\,2}(\Gamma) \oplus \overline{S_{\cancel{k}\,2}(\Gamma)}$$

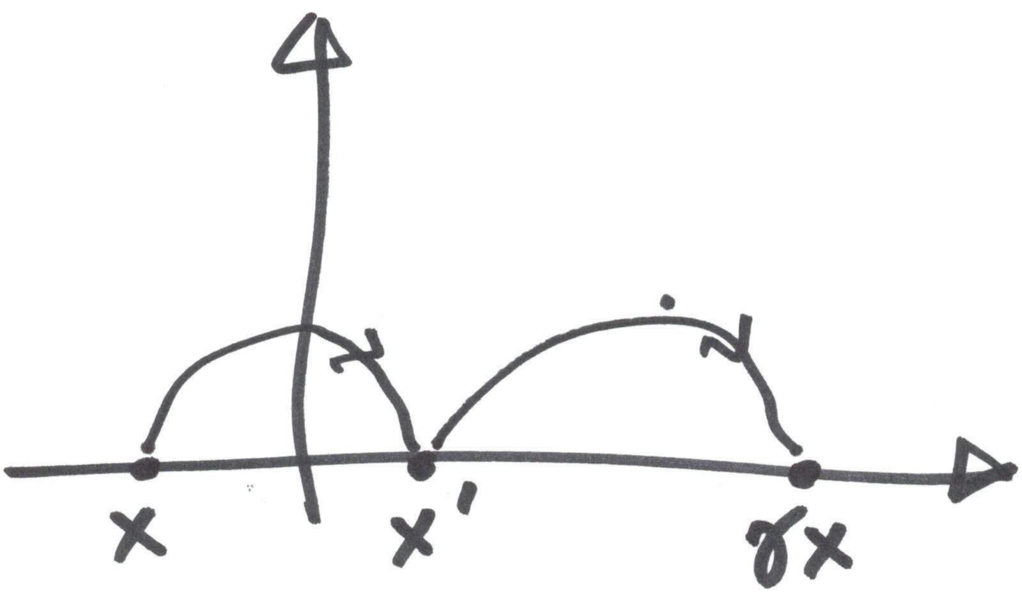
← ANTI-HOLOMORPHIC FORMS.

MODULAR FORMS ARE DUAL TO HOMOLOGY.

$$H_1(X(\Gamma)) \simeq S_2(\Gamma) \oplus \overline{S_2(\Gamma)}$$

$$\left\{ \begin{matrix} \text{LOOPS IN} \\ X(\Gamma) \end{matrix} \right\} / N \hookrightarrow \left\{ \begin{matrix} \text{PATHS IN} \\ X(\Gamma) \\ \text{WITH ENDPOINTS} \\ \text{IN } \Delta \end{matrix} \right\}$$





PROP: $H_1(\mathcal{H}^{2*}, \mathbb{P}^1(\mathbb{Q}); \mathbb{Q})$

$$\simeq \frac{\mathbb{Q}\langle \{x,y\} : x,y \in \mathbb{P}^1(\mathbb{Q})\rangle}{\langle \{x,y\}+\{y,z\}+\{z,x\} : x,y,z \in \mathbb{P}^1(\mathbb{Q})\rangle} =: \mathcal{M}_2$$

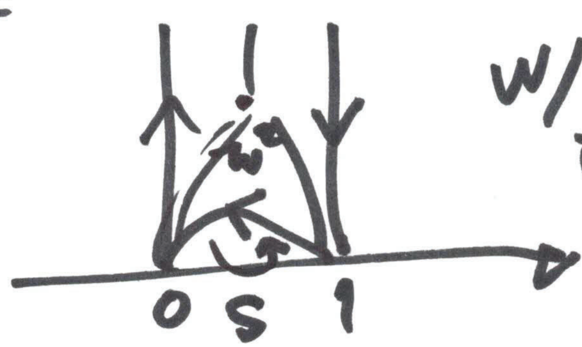
$$\simeq \frac{R}{R(1+S)+R(1+ST+(ST)^2)}\{0,\infty\}$$

$S^2=-1 \quad (ST)^3=-1.$

$R := \mathbb{Q}[PSL_2(\mathbb{Z})$

$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ ACTS TRIVIALLY

PROOF: TRIANGULATE $\mathcal{H}^2$



W/ $PSL_2(\mathbb{Z})$ TRANSLATES

<u>COR</u>: $H_1(X(\Gamma), \Delta; \mathbb{Q}) = \mathcal{M}_{2,\Gamma}$

COINVARIANTS

$$\mathcal{M}_{2,\Gamma} = \mathcal{M}_2 \Big/ \langle \gamma \cdot \{x,y\} - \{x,y\} : \{x,y\},\ \gamma \in \Gamma \rangle .$$

$$\cong \frac{\displaystyle\bigoplus_{\Gamma\delta \in \Gamma \backslash PSL_2(\mathbb{Z})} \mathbb{Q}\langle \delta \rangle \cdot \{0,\infty\}}{\langle \delta + \delta S,\ \delta + \delta S T + \delta (ST)^2 : \delta \in \Gamma\backslash PSL_2(\mathbb{Z}) \rangle}.$$

HECKE OPERATORS: EQUIVARIANT W.R.T. INTEGRATION

$$T_p\{x,y\} = \sum_{\alpha \in \Theta_p} \{\alpha x, \alpha y\}.$$

REDUCTION: WRITE $\{x,y\} = \{0,y\} - \{0,x\}$

$$\underset{x = \frac{a}{c}}{\{0,x\}} = \sum_j \left\{ \frac{a_j}{c_j}, \frac{a_{j+1}}{c_{j+1}} \right\} = \sum_j \pm \underset{\in SL_2(\mathbb{Z})}{\delta_j} \cdot \{0,\infty\}.$$