PAWS 2024: LOCAL FIELDS
PROBLEM SET 4

JACKSYN BAKEBERG, THOMAS BROWNING, ANNA DIETRICH, SIDNEY WASHBURN

The goal for the exercises in Problem Set 4 is to give you some time to practice with the tools and ideas we used to classify non-archimedean fields. The exercises are organized into beginner, intermediate, and advanced sections. We also included a bonus section that has some problems from commutative algebra that are related to the proof of Proposition 4.3.7 in the course notes. As always, choose the problems that feel the most interesting to you!

Beginner

Problem 1. Let $L = K(\alpha)$ be a finite extension of fields of degree n. Show that if α has minimal polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $N_{L/K}(\alpha) = (-1)^n a_0$.
Hint: Use the basis $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$.

Problem 2. Let K be a field of characteristic 0.

(1) Show that if m is odd, then $K(\zeta_{2m}) = K(\zeta_m)$.
(2) Show that $K(\zeta_3) = K(\sqrt{-3})$ and $K(\zeta_4) = K(\sqrt{-1})$.
(3) What is the minimal polynomial of ζ_5 over $\mathbb{Q}$?
(4) What is the degree $[\mathbb{Q}(\zeta_5) : \mathbb{Q}]$ of the extension $\mathbb{Q}(\zeta_5)/\mathbb{Q}$?

Problem 3. Consider the fields $K = \mathbb{Q}_p$ and $L = \mathbb{Q}_p(\sqrt[3]{p})$.

(1) Calculate $v_L(\sqrt[3]{p})$, where $v_L(\alpha) = \frac{1}{n} v_K(N_{L/K}(\alpha))$.
(2) Calculate the normalized valuation $v_{\mathfrak{p}_L}(\sqrt[3]{p})$.
Hint: See the conventions warning on page 25 of the course notes.
(3) Use this to show that the normalization of an extension of a valuation is not necessarily the extension of the normalization of the valuation.

Problem 4. Let A be a ring, and B be a subring of A. An element $a \in A$ is *integral* over B if a is a root of a monic polynomial over B. The subring of A containing all elements integral over B is called the *integral closure* of B in A. If A is equal to this integral closure, then A is called *integral* over B.

(1) Let R, S, and T be rings such that $R \subseteq S \subseteq T$. If T is integral over R, prove that S is integral over R, and that T is integral over S.
(2) Suppose that $R \subseteq S$ are integral domains, and S is integral over R. Show that R is a field if and only if S is a field.
(3) Let $R \subseteq S$ be rings such that $S \setminus R$ is closed under multiplication. Show that R is integrally closed in S.

Problem 5.

(1) Show that $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{2}+\sqrt{3}$ are integral over $\mathbb{Z}$ by writing explicit monic polynomials that they satisfy.
(2) The *trace* map $\mathrm{Tr}_{L/K} : L \to K$ is the trace of the matrix corresponding to the K-linear map on L defined by multiplication by $\alpha \in L$. Let $L = \mathbb{Q}[\sqrt{d}]$. Suppose that $\alpha = a + b\sqrt{d} \in \mathscr{O}_L$, where $a, b \in \mathbb{Q}$. Compute $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha)$ and $\mathrm{N}_{K/\mathbb{Q}}(\alpha)$.
(3) Show that $\mathbb{Z}[\sqrt{-3}]$ is integral over $\mathbb{Z}$, but it is *not* the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{-3})$. What is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{-3})$?

Problem 6. Let K be a non-archimedean local field and let L/K be a degree n extension, endowed with the absolute value $|\gamma|_L = |N_{L/K}(\gamma)|_K^{1/n}$. Show that $|\gamma|_L \leq 1$ if and only if $N_{L/K}(\gamma)$ is in the integral closure $\mathcal{A}$ of $\mathcal{O}_K$ in L.

Problem 7. For any natural number n, prove that $\dfrac{\nu_p(n)}{n-1} \leq \dfrac{1}{p-1}$ and $\nu_p(n!) = \frac{1}{p-1}\sum_{i=0}^r a_i(p^i - 1)$, where $n = \sum_{i=0}^r a_i p^i$ with $0 \leq a_i \leq p-1$.

Problem 8. Prove that if $\nu_p(x) > 0$, then $\left|\dfrac{x^n}{n}\right|_p \to 0$ as $n \to \infty$.

Intermediate

Problem 9. Prove that $U^{(1)} := \{x \in K \mid |x-1| < 1\}$ has the structure of a $\mathbb{Z}_p$-module by defining for $a \in \mathbb{Z}_p$ and $x \in U^{(1)}$, define $x^a := \lim_{n\to\infty} x^{a_n}$, where a_n is any sequence of rational integers converging to a.

Problem 10. Describe the structure of the multiplicative groups $\mathbb{Q}_3^\times$, $\mathbb{Q}_3(i)^\times$ and $\mathbb{Q}_3(\sqrt{3})^\times$.

Problem 11. Classify all continuous group homomorphisms $\chi\colon \mathbb{Q}_p^\times \to \mathbb{C}^\times$.

Problem 12.

(1) Define a measure $\mu_\times$ on $\mathbb{Q}_p^\times$ by

$$d\mu_\times(\alpha) = \frac{d\mu(\alpha)}{|\alpha|}$$

where μ is the measure on $\mathbb{Q}_p$ from Problem Set 3, restricted to the subset $\mathbb{Q}_p^\times \subset \mathbb{Q}_p$.

Show that $\mu_\times$ is multiplication-invariant, i.e.

$$\mu_\times(xA) = \mu_\times(A).$$

(2) Compute $\mu_\times(\mathbb{Z}_p^\times)$ and $\mu_\times(1 + p^n\mathbb{Z}_p)$.

(3) For a function $f : \mathbb{Q}_p \to \mathbb{C}$ and a continuous group homomorphism $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, define the zeta-function

$$\zeta(f, \chi) := \int_{\mathbb{Q}_p^\times} f(\alpha)\chi(\alpha)d\mu_\times(\alpha).$$

Compute $\zeta(f_n, \chi_s)$ where $f_n(x) = \mathbb{I}_{p^n\mathbb{Z}_p}$ and $\chi_s(x) = |x|^s$ for $s \in \mathbb{C}$.

(4) Compute the ratio

$$\gamma := \frac{\zeta(f_n, \chi_s)}{\zeta(\widehat{f_n}, \chi_{1-s})}.$$

where $\widehat{f_n}$ is the Fourier transform of f_n.

On which parameters (e.g. n, s) does γ depend?

Advanced

Problem 13. In this problem we will show that $\mathbb{F}_p((t))^\times \cong \mathbb{Z} \oplus \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}_p^{\mathbb{N}}$.

(1) Show that $g_n(a) = (1+t^n)^a$ defines a function $g_n\colon \mathbb{Z}_p \to U^{(n)}$.

(2) Show that $U^{(np^s)} = g_n(p^s\mathbb{Z}_p)U^{(np^s+1)}$.

(3) Show that $a \notin p\mathbb{Z}_p \iff g_n(p^s a) \notin U^{(np^s+1)}$.

(4) Show that

$$g = \prod_{\gcd(n,p)=1} g_n$$

defines a function $g\colon \mathbb{Z}_p^{\mathbb{N}} \to U^{(1)}$.

(5) Use part (1) to show that g is surjective by showing that $g(A)$ is dense and compact.

(6) Use part (2) to show that g is injective and deduce that $U^{(1)} \cong \mathbb{Z}_p^{\mathbb{N}}$.

(7) Conclude that $\mathbb{F}_p((t))^\times \cong \mathbb{Z} \oplus \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}_p^{\mathbb{N}}$.

Bonus: Commutative Algebra

Problem 14. A *short exact sequence* of groups is a series of maps

$$1 \longrightarrow G \xrightarrow{\varphi} H \xrightarrow{\psi} F \longrightarrow 1,$$

where φ is injective, ψ is surjective, and $\mathrm{Im}(\varphi) = \ker(\psi)$.

(1) Show that if there is a short exact sequence as above, then $F \cong H/G$.
(2) Suppose that G, H, and F are abelian. Show that the following are equivalent:
   (a) There is a homomorphism $\theta : H \to G$ such that $\theta \circ \varphi$ is the identity on G.
   (b) There is an isomorphism $H \cong G \oplus F$.
   Hint: Consider the maps $G \to G \oplus F$ and $G \oplus F \to F$ given by $g \mapsto (g, 0)$ and $(g, f) \mapsto f$, respectively.

Problem 15. (Snake Lemma) Let R be a commutative ring (e.g., $\mathbb{Z}_p$).

(1) Show that if $f\colon M \to N$ is an R-module homomorphism then the sequence of R-modules

$$0 \longrightarrow \ker f \longrightarrow M \xrightarrow{f} N \longrightarrow \operatorname{coker} f \longrightarrow 0$$

is exact where $\operatorname{coker} f = N/\operatorname{im} f$.
(2) Show that the blue commutative diagram of R-modules induces the red homomorphisms of R-modules

$$\begin{tikzcd}
0 \arrow[r] & \ker\alpha \arrow[r] \arrow[d] & A \arrow[r, "\alpha"] \arrow[d, "f"] & A' \arrow[r] \arrow[d, "f'"] & \operatorname{coker}\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\beta \arrow[r] & B \arrow[r, "\beta"] & B' \arrow[r] & \operatorname{coker}\beta \arrow[r] & 0
\end{tikzcd}$$

(3) Show that the blue commutative diagrams of R-modules with exact rows induce the red exact sequences of R-modules

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & \ker\alpha \arrow[r] \arrow[d] & \ker\beta \arrow[r] \arrow[d] & \ker\gamma \arrow[d] \\
 & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
0 \arrow[r, "g"] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "g"] \arrow[d, "\gamma"] & 0 \\
A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] \\
\operatorname{coker}\alpha \arrow[r] \arrow[d] & \operatorname{coker}\beta \arrow[r] \arrow[d] & \operatorname{coker}\gamma \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

(4) Show that the blue commutative diagram of R-modules with exact rows induces the red exact sequence of R-modules

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \ker f \arrow[r] & \ker\alpha \arrow[r] \arrow[d] & \ker\beta \arrow[r] \arrow[d] & \ker\gamma \arrow[d] \arrow[ddll, out=0, in=180, "\delta"'] \\
 & & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
 & 0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] \\
 & & \operatorname{coker}\alpha \arrow[r] \arrow[d] & \operatorname{coker}\beta \arrow[r] \arrow[d] & \operatorname{coker}\gamma \arrow[r] \arrow[d] & \operatorname{coker} g' \arrow[r] & 0 \\
 & & 0 & 0 & 0
\end{tikzcd}$$