PAWS 2024: LOCAL FIELDS
PROBLEM SET 2

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The goal for Problem Set 2 is to practice what we have been learning about completions. The exercises are organized into beginner, intermediate, and advanced levels. If this is your first time thinking about completions or topology, the beginner and intermediate problems might be a good place to start. These exercises are meant for your enjoyment and learning, so make sure to work on the problems that interest you the most!

Beginner

Problem 1. Compute the t-adic expansion of $\frac{1}{t^2+t+1} \in \mathbb{F}_p((t))$.

Problem 2. Prove that $x \in \mathbb{Q}_p$ is rational if and only if its p-adic expansion is eventually periodic.

Problem 3. For which Laurent series $f(t) \in \mathbb{F}_q((t))$ is the t-adic expansion eventually periodic?

Problem 4. In Lecture 3, we will give an explicit description of the set of squares

$$(\mathbb{Q}_p^\times)^2 = \{a^2 : a \in \mathbb{Q}_p^\times\} \subseteq \mathbb{Q}_p^\times.$$

(1) Give an explicit description of the set of squares

$$(\mathbb{F}_2((t))^\times)^2 = \{f^2 : f \in \mathbb{F}_2((t))^\times\} \subseteq \mathbb{F}_2((t))^\times$$

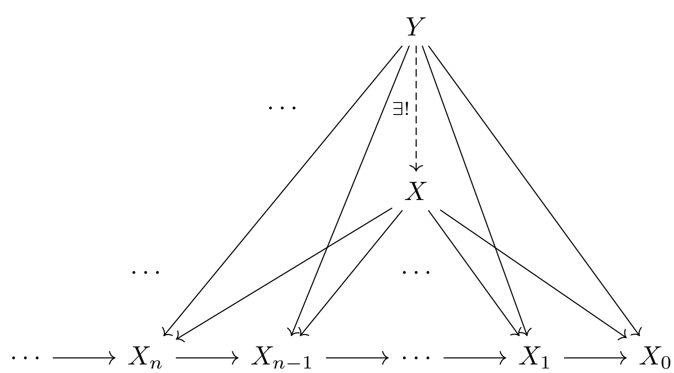
in terms of the Laurent series expansion of f.

Hint: Since the characteristic is 2, we know that $(f+g)^2 = f^2 + g^2$.

(2) Is the set of squares in $\mathbb{F}_2((t))^\times$ open?

(3) Is the set of squares in $\mathbb{F}_2((t))^\times$ closed?

Problem 5. A *projective system* is a sequence of objects $X_0, X_1, \ldots$ (e.g. sets, topological spaces, groups, etc.) together with morphisms $\varphi_{i,j} : X_i \to X_j$ for every $j \leq i$ so that $\varphi_{i,i} = \mathrm{id}$ and if $i \leq j \leq k$, then $\varphi_{i,k} = \varphi_{i,j} \circ \varphi_{j,k}$. The *projective/inverse limit* of the projective system (X_i), if it exists, is an object X together with maps $X \to X_i$ for each i so that the relevant diagrams commute and if Y is any other object together with such maps $Y \to X_i$, then there exists a unique map $Y \to X$ making all the relevant diagrams commute. We write $X = \varprojlim X_n$.



(1) Let $\cdots \subseteq X_n \subseteq X_{n-1} \subseteq \cdots X_1 \subseteq X_0$ be a decreasing sequence of subsets. Let $\varphi_{i,j}$ be the inclusion map from $X_i \hookrightarrow X_j$ for $j \leq i$. Do the X_i form a projective system? If they do, what is the projective limit?

(2) Let $\cdots \to X_n \to X_{n-1} \to \cdots \to X_1$ be a projective system of sets. Determine the projective limit of the X_n. (If you're stuck, check example 2.2.7 in the notes)
(3) Suppose that $\cdots \to D_n \to D_{n-1} \to \cdots D_1$ forms a projective system of sets. Let $D = \varprojlim D_n$, the projective limit. Prove if each D_n is finite and non-empty, then D is non-empty.
(4) Let $f \in \mathbb{Z}_p[x]$. Prove that f has a root in $\mathbb{Z}_p$ if and only if f has a root in $\mathbb{Z}/p^n$ for every $n > 0$.
(5) Prove the generalization: Let $\{f_i\}$ be any set of polynomials in $\mathbb{Z}_p[x_1, ..., x_n]$. Then the f_i have a common zero in $\mathbb{Z}_p^m$ if and only if the f_i have a common zero in $(\mathbb{Z}/p^n)^m$ for all $n > 0$.

Intermediate

Problem 6. Prove the weaker version of Newton's Lemma: If $f \in \mathbb{Z}_p[x]$ has a simple root a_0 modulo p, then f has a unique root $a \in \mathbb{Z}_p$ so that $a \equiv a_0$ modulo p. (Hint: Construct a Cauchy sequence converging to the root using the "Taylor expansion" around a point: $f(x+h) = f(x) + hf'(x) + \cdots$.)

Problem 7.

(1) Show that $\mathbb{Z}_p/(p^2)$ is not isomorphic to $\mathbb{F}_p[[t]]/(t^2)$.
(2) Find a prime p and finite extension $K/\mathbb{Q}_p$ with the property that $\mathcal{O}/\mathfrak{p}^2$ is isomorphic to $\mathbb{F}_p[[t]]/(t^2)$. *Hint*: Use one of the valuations on $\mathbb{Q}(i)$ that we have seen so far.
(3) Is there a finite extension $K/\mathbb{Q}_p$ such that $\mathcal{O}/\mathfrak{p}^n \simeq \mathbb{F}_p[[t]]/(t^n)$ for all n?

Problem 8.

(1) Let $K/\mathbb{Q}_p$ be a valued field extension so that the absolute value on K extends that of $\mathbb{Q}_p$. Suppose that $x \in K$ has following property: there exists n integer sequences $a_0(j)$, $a_2(j)$,..., $a_{n-1}(j)$ ($j \in \mathbb{N}$) so that $|x^n + a_{n-1}(j)x^{n-1} + \cdots + a_0(j)| \to 0$ as $j \to \infty$. Prove that $|x| \leq 1$.
(2) Prove that x is algebraic over $\mathbb{Q}_p$ with degree no greater than n. (i.e. x is a root of a degree $\leq n$ polynomial with coefficients in $\mathbb{Q}_p$)

Problem 9.

(1) Show that $\mathbb{Z}_p$ is sequentially compact, i.e. every sequence has a convergent subsequence.
(2) Show that this is equivalent to $\mathbb{Z}_p$ being compact, i.e. every open cover has a finite subcover.
(3) Show furthermore that this is equivalent to $\mathbb{Z}_p$ being complete and *totally bounded* (for every $\epsilon > 0$, there exists a finite collection of open balls of radius ϵ centered in $\mathbb{Z}_p$ that covers $\mathbb{Z}_p$).

Problem 10. Suppose that X and Y are topological spaces. The *product topology* on $X \times Y$ consists of the subsets of $X \times Y$ generated by the base

$$\mathcal{B} = \{U \times V : U \text{ is open in } X, \text{ and } V \text{ is open in Y}\}.$$

For an infinite product of topological spaces X_i, the product topology is generated by

$$\mathcal{B} = \left\{ \prod_i U_i : U_i \text{ is open in } X_i, \text{ and } U_i = X_i \text{ for all but finitely many } i \right\}.$$

(1) Let $\pi_i : X \to X_i$ be the projection map, and let Y be an arbitrary topological space with a map $f : Y \to X$. Show that under the product topology, f is continuous if and only if $\pi_i \circ f$ is continuous.
(2) Let $\{X_i\}$ be a finite collection of discrete topological spaces. Show that the product $\prod_i X_i$ is discrete.
(3) Give an example to show that an infinite product $\prod_i X_i$ of discrete topological spaces may not be discrete.

Advanced

Problem 11. Let F be a field that is complete with respect to an archimedean absolute value $|\cdot|$. The goal for this problem is to prove Ostrowki's theorem that F is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

(1) In Problem 6 of Problem Set 1, we proved that an absolute value $|\cdot|$ is non-archimedean if and only if the set of values it takes on the integers is bounded. Use this to show that F is characteristic 0.
(2) Use part (1) to argue that $\mathbb{Q} \subset F$. We will denote the topological (not algebraic) closure of $\mathbb{Q}$ in F by $\overline{\mathbb{Q}}$. Show that $\overline{\mathbb{Q}}$ is isomorphic to $\mathbb{R}$.

Thus, F is a field extension of $\mathbb{R}$. A *semi-valuation* is a multiplicative valuation where $\ker(|\cdot|)$ may be larger than 0.

(3) We will show that any archimedean semi-valuation $|\cdot|$ on $\mathbb{R}[t]$ is not a valuation, i.e. that the semi-valuation has a non-trivial kernel.
   (a) By Ostrowski's theorem, there exists some $s \in (0,1]$ such that $|a| = |a|_\infty^s$ for all $a \in \mathbb{R}$. Use this to show that $f(a) = |t-a|$ obtains a global minimum m_1, which we may assume is greater than 0. Similarly show that the quadratic $|t^2+at+b|$ attains its minimum m_2.
   (b) Make a change of variable so that $|f(t)| \geq 1$ for all monic polynomials $f(t)$, and $|f_0| = 1$ for some monic polynomial $f_0(t)$ of degree 1 or 2. Make a further change of variable so that $f_0(t) = t$ or $f_0(t) = t^2 + a^2$.
   (c) First consider the linear case where $f_0(t) = t$ and $|t| = 1$. For any $a \in \mathbb{R}$ with $|a| < 1$, factor $|t^n - a^n|$ to show that $|t-a| = 1$. Thus, if $|t| = 1$, then $|t-a| = 1$ whenever $|a| < 1$. More generally, show that if $|t-a| = 1$, then $|t-b| = 1$ whenever $|a-b| < 1$. Explain why this leads to a contradiction.
   (d) Now consider the quadratic case where $f_0(t) = t^2 + a^2$. Similar to the previous part, show that $|t^2+b^2| = 1$ whenever $|a-b| < 1$, which leads to a contradiction.

(4) Use Part (3) to show that F cannot be a transcendental extension of $\mathbb{R}$, and therefore F is $\mathbb{R}$ or $\mathbb{C}$.

Problem 12. Prove that if K is complete with respect to an absolute value, and V is an n-dimensional normed vector space over K, then for any basis $b_1, \ldots b_n$ of V, the maximum norm

$$|b_1x_1 + \ldots + b_nx_n|_{\max} = \max\{|x_1|, \ldots, |x_n|\}$$

is equivalent to the given norm on V. In particular, V is complete, and we have a homeomorphism $K^n \simeq V$ given by

$$(x_1, \ldots, x_n) \mapsto b_1x_1 + \ldots + b_nx_n.$$

Hint: Find constants $C_1, C_2 > 0$ such that $C_1|x|_{\max} \leq |x| \leq C_2|x|_{\max}$ for all $x \in V$.

Problem 13. Let $K/\mathbb{Q}_p$ be a valued field extension so that the absolute value on K extends that of $\mathbb{Q}_p$. An *nth root of unity* is a value $x \in K$ such that $x^n = 1$. Show that if K contains infinitely many roots of unity, then K is not locally compact.

Hint: Pick a sequence of distinct roots of unity and try to find a convergent subsequence. Relevant calculations can be found in Example 5.3.4 from the notes.