PAWS 2024: LOCAL FIELDS
PROBLEM SET 1

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The goal for Problem Set 1 is to gain some more familiarity with archimedean and non-archimedean absolute values. The exercises are organized into beginner, intermediate, and advanced levels. If this is your first time seeing valuations, we suggest focusing on the beginner and intermediate problems to start. However, you're welcome and encouraged to tackle any problems that catch your interest!

Beginner

Problem 1. Let $|\cdot|$ be a non-archimedean absolute value on a field K.

(1) Show that if $|x| < |y|$, then $|x+y| = |y|$.
(2) Give an example where $|x| = |y|$ and $|x+y| = |y|$.
(3) Give an example where $|x| = |y|$ and $|x+y| < |y|$.

Problem 2. Recall that an infinite series $\sum_{n=1}^{\infty} x_n$ is said to converge if the sequence of partial sums $\sum_{n=1}^{N} x_n$ converges as $N \to \infty$.

(1) Show that in $\mathbb{R}$ and $\mathbb{Q}_p$, a series converges if and only if the sequence of partial sums is a Cauchy sequence.
(2) Prove that in $\mathbb{Q}_p$, a series $\sum_{n=1}^{\infty} x_n$ converges if and only if $x_n \to 0$ as $n \to \infty$.
(3) In $\mathbb{R}$, is it true that a series $\sum_n x_n$ converges if and only if $x_n \to 0$? Prove or disprove.
(4) Prove that a power series $\sum_n a_n x^n$ coverges when $|x| < R$, where $R = 1/\limsup_{n\to\infty} |a_n|^{1/n}$. Prove it diverges is $|x| > R$. (Note that your proof should work in $\mathbb{R}$)
(5) Prove that in $\mathbb{Q}_p$, if a power series $S(x) = \sum_n a_n x^n$ converges for some $|x| = R$, then S converges at every point of absolute value R. Is the same statement true in $\mathbb{R}$ or $\mathbb{C}$?

Problem 3. Show that the only possible absolute value on the finite field $\mathbb{F}_q$ is the trivial absolute value.

Problem 4. Show that for any prime p, the sequence $(n!)_{n\geq 0}$ converges to 0 under the topology induced by the p-adic absolute value.

Problem 5. Examples 1.1.7 and 1.1.8 in the lecture notes define absolute values on $\mathbb{Q}(i)$ that agree with $|\cdot|_3$ and $|\cdot|_5$ when restricted to $\mathbb{Q}$. Define an absolute value on $\mathbb{Q}(i)$ that agrees with $|\cdot|_2$ when restricted to $\mathbb{Q}$.

Intermediate

Problem 6. We will show that an absolute value $|\cdot|$ on $\mathbb{Q}$ is non-archimedean if and only if $|n| \leq 1$ for all $n \in \mathbb{Z}$.

(1) Suppose that $|\cdot|$ is any absolute value on $\mathbb{Q}$. Show that $|1| = 1$ and $|-1| = 1$.
(2) Use induction to show that if $|\cdot|$ is non-archimedean, then $|n| \leq 1$ for all $n \in \mathbb{Z}$.
(3) Conversely, suppose that $|\cdot|$ is an absolute value on $\mathbb{Q}$ such that $|n| \leq 1$ for all $n \in \mathbb{Z}$. Show that $|\cdot|$ is non-archimedean.

    Hint: Apply the binomial expansion to $(x+y)^n$.

Problem 7. Let $(K, |\cdot|)$ be a valued field. Show that the completion $(\widehat{K}, |\cdot|)$ is complete and is unique. (Exercise 1.3.8 from the notes).

Problem 8. It is possible to express $-1/15 = \sum_{v=m}^{\infty} a_v p^v$, where $a_v \in \{0, 1, 2, 3, 4\}$?

(1) What is the minimal n so that $a_n \neq 0$?
(2) Find the first five coefficients a_n.

Problem 9.

(1) Prove the *product formula*: for every rational number $a \neq 0$, we have
$$\prod_p |a|_p = 1,$$
where p varies over all prime numbers as well as the symbol ∞.
(2) Prove the product formula for $\mathbb{F}_q(t)$: for every rational function $a(t) \in \mathbb{F}_q(t)$,
$$\prod_{p(t)} |a(t)|_{p(t)} = 1,$$
where $p(t)$ varies over all monic irreducible polynomials as well as the symbol ∞.

Problem 10. The goal of this exercise is to prove *Ostrowski's theorem*: every non-trivial absolute value $|\cdot|$ on $\mathbb{Q}$ is equivalent to either a p-adic valuation $|\cdot|_p$ or the standard absolute value $|\cdot|_\infty$.

We consider two cases:

Case (i): $|\cdot|$ is non-archimedean.

(1) In Problem 6, we showed that $|n| \leq 1$ for all integers n. Show that there is a prime number p such that $|p| < 1$.
Hint: Unique factorization.
(2) Show that $\mathfrak{p} := \{x \in \mathbb{Z} \mid \ |x| < 1\}$ is an ideal of $\mathbb{Z}$ and find a generator for this ideal.
(3) Show that $|\cdot|$ is equivalent to $|\cdot|_p$.

Case (ii): $|\cdot|$ is archimedean.

(1) Show that for every $m, n > 1$,
$$|m|^{1/\log m} = |n|^{1/\log n}.$$
Hint: we can write $m = a_0 + a_1 n + \cdots + a_r n^r$ where $a_i \in \{0, 1, \ldots, n-1\}$ and $n^r \leq m$. Use this and the triangle inequality to bound $|m|$ in terms of $|n|$, $\log m$, and $\log n$. Replace m with m^k, take k-th roots, and let $k \to \infty$ to get $|m| \leq |n|^{\log m/\log n}$. Repeat with m and n switched.
(2) Show that $|\cdot|$ is equivalent to $|\cdot|_\infty$.

Problem 11. Here we will show an analogue of Ostrowski's theorem for the function field $\mathbb{F}_q(t)$: every non-trivial absolute value $|\cdot|$ on $\mathbb{F}_q(t)$ is equivalent to either a $p(t)$-adic valuation $|\cdot|_{p(t)}$ or the degree absolute value $|\cdot|_\infty$.

We consider two cases:

Case (i): $|t| > 1$.

(1) First show that for a non-zero polynomial $f(t) = a_0 + \ldots + a_n x^n$,
$$|f(t)| = |t|^{\deg f}.$$
Hint: Problem 3 may help.
(2) Use this to show that in this case, $|\cdot| = |\cdot|_\infty$.

Case (ii): $|t| \leq 1$.

(1) Show that for a non-zero polynomial $f(t) = a_0 + \ldots + a_n x^n$,
$$|f(t)| \leq \max |a_i t^i| \leq 1.$$
(2) Make an argument that there must be a monic, non-constant polynomial $p(t)$ for which $|p(t)| < 1$, and that $p(t)$ is irreducible. Suppose that $p(t)$ is the polynomial of least degree with this property.
(3) Note that it suffices to show that $|\cdot|$ is equivalent to $|\cdot|_{p(t)}$ on polynomials. Write the polynomial $f(t)$ as $p(t)^m g(t)$, where $p(t)$ does not divide $g(t)$. Show that $|g(t)| = 1$.
Hint: Polynomial division algorithm.

Advanced

Problem 12. Let p be an odd prime. Determine when $\mathbb{Q}_p$ contains $\sqrt{-1}$ in terms of p. How is this related to the prime factorization of $p \in \mathbb{Z}[i]$? (Recall that $(-1/p) = (-1)^{(p-1)/2}$, where (a/p) is the Legendre symbol.)

Problem 13. Let $P \in \mathbb{P}^N(\mathbb{Q})$ be a point with homogeneous coordinates $[x_0 : \ldots : x_n]$, and let M denote the set of equivalence classes of absolute values on $\mathbb{Q}$. Then, the *height of* P is

$$H_{\mathbb{Q}}(P) = \prod_{v \in M} \max\{|x_0|_v, \ldots, |x_n|_v\}.$$

(1) Show that for any C, the set

$$\{P \in \mathbb{P}^N(\mathbb{Q}) : H_{\mathbb{Q}}(P) \leq C\}$$

is finite.

Hint: Problem 6 may help. What does the height function over $\mathbb{Q}$ actually look like?

(2) Can you think of a good upper bound dependent on the choice of C for the size of the set in Part (1)?

(3) **Challenge:** Let K be a finite extension of $\mathbb{Q}$ such as $\mathbb{Q}(i)$, and let M_K be the set of absolute values on K. The *height of* P *relative to* K is

$$H_K(P) = \prod_{v \in M_K} \max\{|x_0|_v, \ldots, |x_n|_v\}^{[K_v:\mathbb{Q}_v]}.$$

Does the result from Part (1) still hold for H_K?

Remark: For more information about height functions and how they are used in arithmetic geometry, see Chapter 3 of *The Arithmetic of Dynamical Systems* or Chapter 8 of *The Arithmetic of Elliptic Curves* by Silverman.

Problem 14. Let $|\cdot|_1, \ldots, |\cdot|_n$ be pairwise inequivalent nontrivial absolute values on a field K. Let $a_1, \ldots, a_n \in K$, and let $\varepsilon > 0$. Show that there exists an $x \in K$ such that $|x - a_k|_k < \varepsilon$ for all $k = 1, \ldots, n$.

(1) Hint: First prove that there exists a so that $|a|_1 > 1$ and $|a|_i < 1$ for $i > 1$. (Induct on n and exploit the limiting properties of x^r and $x^r/(1+x^r)$ as $r \to \infty$).

(2) Next prove that there exists a close to 1 with respect to $|\cdot|_1$ and close to 0 for $|\cdot|_i$, $i > 1$. (Again, $x^r/(1+x^r)$ is useful).

(3) Prove the original statement.