PAWS Root Systems: PROBLEM SET 5

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Question 1: This problem deals with p-adic integers and p-adic numbers.

(1) Consider the sequence of integers

$$4, 34, 334, 3334, ...$$

Is this a Cauchy sequence with respect to the 5-adic topology on $\mathbb{Q}$? If so, what is the limit?

(2) Show that every series of the form

$$a_{-n}p^{-n} + \cdots + a_0 + a_1 p + \cdots + a_m p^m + \cdots, \quad \text{with } 0 \leq a_i < p \text{ for all } i,$$

has a sequence of partial sums which is a Cauchy sequence with respect to the p-adic topology on $\mathbb{Q}$.

(3) Show that -1 is a square in $\mathbb{Z}_5$. In other words, find a series

$$a_0 + a_1 5 + a_2 5^2 + \cdots, \quad \text{with } a_i \in \{0, 1, 2, 3, 4\} \text{ for all } i,$$

such that

$$(a_0 + a_1 5 + a_2 5^2 + \cdots)^2 + 1 = 0.$$

Question 2:

Let F be a non-archimedean local field and assume that $\mathrm{G} = \mathrm{GSpin}(V)$ is quasi-split (hence you have root datum). Using the root datum for GSpin, show that there is an embedding of the following groups,

$$\mathrm{G}_n \hookrightarrow \mathrm{G}_{n+1} \hookrightarrow \mathrm{G}_{n+2}.$$

One can proceed as follows:

(1) Assume that $n = 2m$ where m is a positive integer. The derived group of GSpin_i is Spin_i. Furthermore, GSpin_i is an almost direct product (has finite intersection) of Spin_i and the connected component of the center. The connected component of the center of G_i is $\{e_0^*(t) : t \in \mathrm{GL}(1)\}$. First, show that for each root subgroup there is an embedding of G_n into G_{n+1} and then into G_{n+2}.

(2) Next, embed the connected component of the center of G_n into G_{n+1} and then into G_{n+2}.

(3) Then, to ensure that these embeddings are well-defined, show that it is well defined on the intersections.

Question 3: Consider the element

$$\mathfrak{q} = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}).$$

Define the $\mathbb{C}$-vector space V using the relation

$$V := \{X \in \mathfrak{gl}_2(\mathbb{C}) \ : \ \mathfrak{q}X\mathfrak{q}^{-1} = qX\}.$$

Define the subgroup $H \subset \mathrm{GL}_2(\mathbb{C})$ using the relation

$$H := \{g \in \mathrm{GL}_2(\mathbb{C}) \ : \ \mathfrak{q}X\mathfrak{q}^{-1} = X\}.$$

The group H acts on V via

$$h \cdot X := hXh^{-1}.$$

(1) Show that $V = \mathbb{C}$. For people with an algebraic geometry bent, $V = \mathbb{A}^1_{\mathbb{C}}$ is an affine variety over $\mathbb{C}$.

(2) GL_2 is associated with the multiplicative root data given by the set of simple roots $\Delta = \{\alpha_1\}$ where

$$\alpha_1 : T_2 \to \mathbb{C}^\times$$
$$\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mapsto \frac{t_1}{t_2}$$

Can you characterize V in the language of roots? (Eg. V is the eigenspace corresponding to the root ... when it attains the value ...)

(3) Show that H is the diagonal torus in GL_2. For people with an algebraic geometry bent, $H = \mathbb{G}_m \times \mathbb{G}_m$.

(4) Show that the H-action on V gives you two orbits, $C_0 = \{0\}$ and $C_1 = \mathbb{C} \setminus \{0\}$. For people with an algebraic geometry bent, convince yourself that C_0 and C_1 are 0- and 1-dimensional varieties over $\mathbb{C}$.

You have computed a geometric version of the local Langlands correspondence for $\mathrm{GL}_2(F)$ for a non-archimedean local field F of residue characteristic $q = p^f$. The two orbits recover (the equivalence class of) the trivial representation $\mathbb{1}_{\mathrm{GL}_2}$ and an infinite-dimensional representation $\mathrm{St}_{\mathrm{GL}_2}$. These representations share an infinitesimal parameter (a semisimple map $\lambda : W_F \to \mathrm{GL}_2(\mathbb{C})$) determined by the matrix $\mathfrak{q}$. Can you guess which orbit corresponds to $\mathbb{1}_{\mathrm{GL}_2}$? For more information, feel free to talk to Mishty.