PAWS Root Systems: PROBLEM SET 4

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Question 1: Following Dr. Emory's explanation for G_2, check that the Cartan matrix for B_ℓ given in the lecture is correct. Use this to re-derive the corresponding Dynkin diagram.

Question 2: In your own words, what parts of our definition of a root system dictate what parts of the structure of a Cartan matrix? What if we did not require condition 4, the integrability condition/Crystallographic condition?

Question 3: The commutator of two square matrices A, B is

$$[A, B] = AB - BA.$$

(1) Compute the commutator of $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 1 & 5 \end{pmatrix}$.

(2) Compute the commutator of $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 7 \\ 0 & 3 \end{pmatrix}$.

(3) One interpretation of the commutator is that it measures the extent to which objects commute. For example, with A and B commuting, $AB = BA$, so the commutator is zero. Prove that if A is a scalar multiple of the identity matrix, then $[A, B] = 0$ for any B.

(4) Analogously to how matrices act on vector spaces, operators like the derivative act on spaces of functions. We know that taking a derivative then multiplying the resulting function by x is different than first multiplying by x and then taking a derivative. To "measure" this non-commutativity, let $\frac{d}{dx}(f) = f'$ and $M_x(f) = x \cdot f$ and compute $[\frac{d}{dx}, M_x]$.

Question 4: Let A be an algebra over a field K. A K-derivation is a K-linear map $\delta : A \to A$ that satisfies

$$\delta(ab) = a\delta(b) + \delta(a)b \quad \text{for all } a, b \in A.$$

(1) Prove that the commutator of two derivations is again a derivation.

(2) Prove that the set of all derivations $\mathcal{D}$ forms a Lie algebra over K.

Figure 1: *Dynkin donuts*. Image credit: Ying Hong Tham.

Question 5: This problem refers to the above figure.

(1) For each Dynkin diagram in the figure above, write down a Cartan matrix associated to it, and identify its associated root system.

(2) Which of the Dynkin diagrams have a nontrivial diagram automorphisms?

Question 6: Draw the Dynkin diagram of D_4. Observe that it is the only connected Dynkin diagram with a diagram automorphism group of size > 2.

Question 7: Prove that the group of diagram automorphisms of a Dynkin diagram associated to a root system $R = (\Phi, V)$ with Weyl group W is isomorphic to the quotient group $\mathrm{Aut}(R)/W$.

Question 8: Prove that a root system is irreducible if and only if its Dynkin diagram is connected.

Question 9: The root system F_4 is the union of the root system B_4 with the vectors

$$(\pm_1 \tfrac{1}{2}, \pm_2 \tfrac{1}{2}, \pm_3 \tfrac{1}{2}, \pm_4 \tfrac{1}{2})$$

for all choices of signs $\pm_1, \dots, \pm_4$. Recall that the $B_\ell = (\Phi, \mathbb{R}^\ell)$ where Φ is the set of vectors with integer coordinates and Euclidean length 1 or $\sqrt{2}$.

(1) Prove that F_4 is an irreducible root system.

(2) Prove that $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ base for F_4, where

$$\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4), \quad \alpha_2 = e_4, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_2 - e_3.$$

(3) Compute the Cartan matrix for F_4 and draw its Dynkin diagram.

Question 10: The E_8 root system is the union of the root system D_8 with the vectors

$$\frac{1}{2}\sum_{j=1}^{8} \pm_j e_j$$

for all choices of signs $\pm_1, \ldots, \pm_8$ with an even number of minus signs. Recall that the $D_\ell = (\Phi, \mathbb{R}^\ell)$ where Φ is the set of vectors with integer coordinates and Euclidean length $\sqrt{2}$.

(1) Prove that E_8 is an irreducible root system.

(2) Check that $\Delta = \{\alpha_1, \ldots, \alpha_8\}$ is a base for E_8, where

$$\alpha_1 = \tfrac{1}{2}(e_1 + e_8 - \sum_{j=2}^{7} e_j), \quad \alpha_2 = e_7 + e_8, \quad \alpha_k = e_{10-k} - e_{11-k} \text{ for } 3 \le k \le 8.$$

(3) Compute the Cartan matrix for E_8 and draw its Dynkin diagram.

(4) For $\ell = 6, 7$, the root system E_ℓ is the subsystem of E_8 generated by $\{\alpha_1, \ldots, \alpha_\ell\}$. Realize the Cartan matrix of E_ℓ as an $\ell \times \ell$ submatrix of the Cartan matrix of E_8. Consequently, realize the Dynkin diagrams for E_6 and E_7 as subgraphs of the Dynkin diagram for E_8.

Question 11: Let $\mathfrak{g}$ be a Lie algebra over a field F.

(1) Show that $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$. This property is called anti-commutativity.

(2) When $\mathrm{char}(F) \neq 2$, show that anti-commutativity is equivalent to the alternating property in the definition of a Lie algebra.

Question 12: In this question, we will think about how the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of 2×2 trace 0 matrices relates to the root system A_1.

(1) Let $\mathfrak{sl}_2(\mathbb{C})$ denote the Lie algebra of 2×2 trace 0 matrices over $\mathbb{C}$. The Lie bracket $[X, Y]$ is given by the commutator $XY - YX$. Convince yourself that this is indeed a Lie algebra.

(2) Convince yourself that this Lie algebra is spanned by the elements

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(3) Consider the map

$$\mathrm{ad} : \mathfrak{sl}_2 \to \mathrm{End}(\mathfrak{sl}_2) \quad \text{given by} \quad X \mapsto [X, -].$$

Show that the subspace spanned by E is an eigenspace of $\mathrm{ad}(H)$ with eigenvalue $\alpha = 2$. Denote the subspace $\mathfrak{g}_\alpha$. Similarly, show that the subspace spanned by F is an eigenspace for $\mathrm{ad}(H)$ with eigenvalue is $-\alpha$. Denote this subspace by $\mathfrak{g}_{-\alpha}$.

(4) Let $\mathfrak{h}$ denote the subspace spanned by H. Observe that

$$\mathfrak{sl}_2 = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}.$$

This is the *root space decomposition* of the Lie algebra of $\mathfrak{sl}_2$ with *Cartan subalgebra* $\mathfrak{h}$. Setting $\Phi = \{\alpha, -\alpha\}$, observe that $(\Phi, \mathbb{R}) \cong A_1$.

Question 13: In this question, we will think about how the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ of 3×3 trace 0 matrices relates to the root system A_2.

(1) Let $\mathfrak{sl}_3(\mathbb{C})$ denote the lie algebra of 3×3 trace 0 matrices over $\mathbb{C}$. The Lie bracket $[X, Y]$ is given by the commutator $XY - YX$. Convince yourself that this is indeed a Lie algebra.

(2) Convince yourself that this Lie algebra is spanned by the elements

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$E_{1,2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ E_{1,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ E_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$F_{2,1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ F_{3,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ F_{3,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(3) Let V be the vector space of real valued linear functions on the real vector space spanned by H_1 and H_2. Identify V with a subspace of $\mathbb{R}^3$ as follows. First, denote by $\overline{e}_i$ the functional which computes the i'th diagonal entry. Consider the linear map $T : \mathbb{R}^3 \to V$ given by $e_i \mapsto \overline{e}_i$. Prove that V may be identified with W, the orthogonal complement of $(1, 1, 1) \in \mathbb{R}^3$, by observing that $T|_W$ is an isomorphism.

(3) Consider the map

$$\mathrm{ad} : \mathfrak{sl}_3 \to \mathrm{End}(\mathfrak{sl}_3) \quad \text{given by} \quad X \mapsto [X, -].$$

For each $X \in \{E_{1,2}, E_{1,3}, E_{2,3}, F_{2,1}, F_{3,1}, F_{3,2}\}$, show that the subspace spanned by X is a simultaneous eigenspace of $\mathrm{ad}(H_1)$ and $\mathrm{ad}(H_2)$ with real eigenvalues, say $\alpha_{1,X}$ and $\alpha_{2,X}$, respectively. Denote this subspace as $\mathfrak{g}_{\alpha_X}$ where $\alpha_X \in V$ is the linear functional given by $\alpha(H_1) = \alpha_{1,X}$ and $\alpha(H_2) = \alpha_{2,X}$. Identifying V with a subspace of $\mathbb{R}^3$ as in part (3), prove that

$$\alpha_{E_{1,2}} = \alpha, \ \alpha_{E_{1,3}} = \alpha + \beta, \ \alpha_{E_{2,3}} = \beta$$
$$\alpha_{F_{2,1}} = -\alpha, \ \alpha_{F_{3,1}} = -(\alpha + \beta), \ \alpha_{F_{3,2}} = -\beta$$

where $\alpha = (1, -1, 0)$ and $\beta = (0, 1, -1)$.

(4) Let $\mathfrak{h}$ denote the subspace spanned by H_1 and H_2. Observe that

$$\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta} \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{-(\alpha+\beta)}.$$

This is the *root space decomposition* of the Lie algebra of $\mathfrak{sl}_2$ with *Cartan subalgebra* $\mathfrak{h}$. Setting $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$, observe that $(\Phi, V) \cong A_2$.

Question 14: For any Lie algebra $\mathfrak{g}$, the *Killing form* is the symmetric bilinear form κ on $\mathfrak{g}$ defined by

$$\kappa(x, y) = \text{trace}(\text{ad}(x) \circ \text{ad}(y)).$$

Here, ad is the map

$$\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g})$$

given by

$$x \mapsto [x, -].$$

(1) Show that for any $x \in \mathfrak{g}$, the operator $\text{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric with respect to κ, i.e.,

$$\kappa([x, y], z) = -\kappa(y, [x, z])$$

for all $y, z \in \mathfrak{g}$.

(2) A Lie algebra is semisimple if and only if the Killing form is *nondegenerate*, i.e. for all nonzero $x \in \mathfrak{g}$, there is some $y \in \mathfrak{g}$ for which $\kappa(x, y) \neq 0$. Read the proof of this fact in Humphreys' *Introduction to Lie Algebras and Representation Theory*.

Question 15: Let $\mathfrak{sl}_n$ denote the Lie algebra consisting of $n \times n$ trace 0 matrices over $\mathbb{R}$ or $\mathbb{C}$. Assume $n = 2, 3$. Show that for $\mathfrak{sl}_n$, the Killing form is given by

$$\kappa(X, Y) = 2n\,\text{trace}(XY).$$