# PAWS Root Systems: PROBLEM SET 3

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Recall the following two definitions from the lecture notes:

**Definition.** *A subset $\Phi^+$ of $\Phi$ is a set of* **positive roots** *for the root system $(\Phi, V)$ if*

- *for each $\alpha \in \Phi$, exactly one of $\alpha$ or $-\alpha$ is contained in $\Phi^+$, and*
- *for any two distinct roots $\alpha, \beta \in \Phi^+$ such that $\alpha+\beta$ is a root, $\alpha+\beta \in \Phi^+$.*

**Definition.** *A subset $\Delta$ of $\Phi$ is a* **base** *for the root system $(\Phi, V)$ if*

- *$\Delta$ is a basis for $V$ as a real vector space, and*
- *each root $\alpha$ can be expressed as a linear combination of elements of $\Delta$ in such a way that all the coefficients of the linear combination are all non-negative or all non-positive.*

**Question 1:** For each of the root systems of rank two, $A_1 \times A_1, A_2, B_2, G_2$,

(1) Identify a set of positive roots for the root system,

(2) Identify a base for the root system,

(3) Draw the fundamental chamber for that root system with respect to that choice of base.

**Question 2:** This exercise clarifies that making a choice of a base $\Delta \subset \Phi$ is equivalent to making a choice a set of positive roots $\Phi^+ \subset \Phi$.

(1) Prove that if $\Phi^+$ is a set of positive roots for the root system $(\Phi, V)$, then the set of elements in $\Phi^+$ which cannot be written as the sum of two elements in $\Phi^+$ is a base for $(\Phi, V)$ (cf. Question 7).

(2) Prove that if $\Delta$ is a base for $(\Phi, V)$ then the set of roots in $\Phi$ that can be written as a non-negative linear combination of elements in $\Delta$ is a set of positive roots for the root system $(\Phi, V)$.

(3) Let $\Delta$ be a base and $\Phi^+$ be a set of positive roots for the root system $(\Phi, V)$. Prove that the following are equivalent:

(a) $\Delta \subset \Phi^+$.

(b) $\Delta$ is the set of elements in $\Phi^+$ which cannot be written as the sum of two elements in $\Phi^+$.

(c) $\Phi^+$ is the set of elements in $\Phi$ that can be written as a non-negative linear combination of elements in $\Delta$.

**Question 3:** If $(\Phi, V)$ is a root system, then for each root $\alpha \in \Phi$, the **coroot** $\alpha^\vee$ is given by

$$\alpha^\vee = 2\frac{\alpha}{(\alpha, \alpha)}.$$

The set of all coroots is denoted by $\Phi^\vee$ and is called the dual root system to $\Phi$.

(1) Given $\alpha = (1, 0)$ and $\beta = (-1/2, \sqrt{3}/2)$, compute $\alpha^\vee$ and $\beta^\vee$.

(2) Verify computationally that $(\alpha^\vee)^\vee = \alpha$ and $(\beta^\vee)^\vee = \beta$.

(3) Prove that for any root $\gamma \in \Phi$, $(\gamma^\vee)^\vee = \gamma$.

**Question 4:** Recall the definition of **isomorphism** of root systems. Use this to show that the root systems $B_2$ and $C_2$ are isomorphic, and the root systems $A_1 \times A_1$ and $D_2$ are isomorphic.

**Question 5:** This exercise has you prove the key proposition that was used to classify root systems of rank two. Suppose $\alpha$ and $\beta$ are roots, $\alpha$ is not a multiple of $\beta$, and $(\alpha, \alpha) \geq (\beta, \beta)$. Prove that one of the following holds:

(1) $(\alpha, \beta) = 0$

(2) $(\alpha, \alpha) = (\beta, \beta)$ and the angle between $\alpha$ and $\beta$ is $\pi/3$ or $2\pi/3$

(3) $(\alpha, \alpha) = 2(\beta, \beta)$ and the angle between $\alpha$ and $\beta$ is $\pi/4$ or $3\pi/4$

(4) $(\alpha, \alpha) = 3(\beta, \beta)$ and the angle between $\alpha$ and $\beta$ is $\pi/6$ or $5\pi/6$

**Question 6:** Prove that if $\alpha$ and $\beta$ are distinct elements of a base $\Delta$ then $(\alpha, \beta) \leq 0$. In other words, prove that the angle formed between any two simple roots is always obtuse or a right angle.

**Question 7:** *In any root system, there always exists a base.*

Think about how you would prove this fact. Then read the proof of Theorem 8.16 in [1][1] Summarize the argument in your own words.

**Question 8:** This exercise will clarify the relationship between various choices of bases of a root system $(\Phi, V)$.

(1) Prove that any two Weyl chambers $C$, $C'$ can be connected by a sequence of chambers $C_0 = C, C_1, \ldots, C_l = C'$ such that $C_i$ is adjacent to $C_{i+1}$.

[1] The book is available at Springer's website or on ResearchGate or at the link below.

(2) Prove that if $C, C'$ are adjacent Weyl chambers separated by hyperplane $H_\alpha$ then $s_\alpha(C) = C'$.

(3) Using the previous two parts, prove that the Weyl group $W$ acts simply transitively on the set of Weyl chambers. Conclude that the number of Weyl chambers is equal to the order of the Weyl group.

(4) Prove that for each Weyl chamber $C$, there exists a unique base $\Delta_C$ for $(\Phi, V)$ such that $C$ is the fundamental Weyl chamber associated to $\Delta_C$.

(5) Let $\Delta$ and $\Delta'$ be two bases for $(\Phi, V)$. Prove that there exists a (unique) Weyl group element $w \in W$ such that $w(\Delta) = \Delta'$. Conclude that the base of a root system is well defined up to Weyl group automorphisms, and moreover, the number of distinct bases is equal to the order of the Weyl group.

**Question 9:** Let $(\Phi, V)$ and $(\Phi', V)$ be two (reduced) root systems in the same Euclidean vector space $V$. Suppose there exists a base for $\Phi$ which is also a base for $\Phi'$. Prove that $\Phi = \Phi'$.

**Question 10:** This problem has you classify the irreducible (reduced) root systems of rank three. Let $(\Phi, V)$ be an irreducible root system of rank three. Let $\Delta = \{\alpha, \beta, \gamma\}$ be a base. Write $\theta_{\alpha,\beta}$, $\theta_{\beta,\gamma}$, $\theta_{\alpha,\gamma}$ to denote be the angles between the given simple roots in the subscripts.

(1) Prove that the irreducibility of $(\Phi, V)$ implies that no two of the angles $\theta_{\alpha,\beta}$, $\theta_{\beta,\gamma}$, $\theta_{\alpha,\gamma}$ can be $90°$.

(2) Prove that $90° \leq \theta_{\alpha,\beta}, \theta_{\beta,\gamma}, \theta_{\alpha,\gamma} < 180°$ and $\theta_{\alpha,\beta} + \theta_{\beta,\gamma} + \theta_{\alpha,\gamma} < 360°$.

(3) Conclude that

- exactly one of the three angles $\theta_{\alpha,\beta}$, $\theta_{\beta,\gamma}$, $\theta_{\alpha,\gamma}$ is $90°$,
- at least one of the other two angles is $120°$,
- and none of the angles are $150°$.

(4) Without loss of generality, assume $\theta_{\alpha,\beta} \leq \theta_{\beta,\gamma} \leq \theta_{\alpha,\gamma}$. Conclude that

$$(\theta_{\alpha,\beta}, \theta_{\beta,\gamma}, \theta_{\alpha,\gamma}) = (90°, 120°, 120°) \text{ or } (90°, 120°, 135°).$$

(5) There are three possibilities:

- Case $A$: $(\theta_{\alpha,\beta}, \theta_{\beta,\gamma}, \theta_{\alpha,\gamma}) = (90°, 120°, 120°)$
- Case $B$: $(\theta_{\alpha,\beta}, \theta_{\beta,\gamma}, \theta_{\alpha,\gamma}) = (90°, 120°, 135°)$ and $(\alpha, \alpha) < (\gamma, \gamma)$
- Case $C$: $(\theta_{\alpha,\beta}, \theta_{\beta,\gamma}, \theta_{\alpha,\gamma}) = (90°, 120°, 135°)$ and $(\alpha, \alpha) > (\gamma, \gamma)$

Verify by inspecting the figures in Section 8.9 of [1] that the root systems $A_3$, $B_3$, $C_3$ fall into the above three cases respectively.

(6) Prove, using the fact that a root system is determined uniquely by any of its bases (see Question 9 ) that any irreducible root system of rank three is isomorphic to $A_3$, $B_3$, or $C_3$.

## References

[1] Brian Hall, *Lie Groups, Lie Algebras, and Representations*, volume 222 of *Graduate Texts in Mathematics*. Springer, Cham, second edition, 2015. An elementary introduction.(link to download)