PAWS Root Systems: PROBLEM SET 2

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Question 1: This question asks us to use the definition of B_2 to determine the characterization of B_3 and then B_ℓ.

(1) Recall the definition of B_2 and compute the Euclidean lengths of the roots as vectors. Notice that there are roots of length 1 and roots of length $\sqrt{2}$. Check that these are all the vectors in $\mathbb{R}^2$ with integer coordinates and with Euclidean length 1 or $\sqrt{2}$.

(2) List all the vectors in $\mathbb{R}^3$ with integer coordinates and with Euclidean length 1. *Hint: there should be 6.* List all the vectors in $\mathbb{R}^3$ with integer coordinates and with Euclidean length $\sqrt{2}$. *Hint: there should be 12.*

(3) Prove that the set of vectors in $\mathbb{R}^3$ with integer coordinates and Euclidean length 1 or $\sqrt{2}$ is a root system. This is the B_3 root system.

(3) Make a guess for a definition of B_ℓ. List all the vectors in B_ℓ.

(4) Check your guess with section 8.10.3 of Hall's *Lie Groups, Lie Algebras, and Representations*. The book is available at Springer's website or on ResearchGate (link to download).

Question 2: With the $A_1 \times A_1$ root system as given in the notes, draw the integral span of Φ. That is, draw

$$m_1(e_1 - e_2) + m_2(-e_1 + e_2) + m_3(e_1 + e_2) + m_4(-e_1 - e_2) \quad \text{with } m_i \in \mathbb{Z}.$$

Question 3: The **dual** of a root system Φ in the finite dimensional real vector space V is the set

$$\Phi^\vee = \{\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} \mid \alpha \in \Phi\}.$$

(1) Prove that $\Phi^\vee$ is a root system in V.

(2) Prove that the Weyl group of $\Phi^\vee$ is isomorphic to the Weyl group of Φ.

(3) Show that $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$.

*(4) Draw a picture of $\Phi^\vee$ in the cases of A_1, A_2, B_2, G_2.

Definition. *Let Δ be a base for a root system Φ. For $\beta \in \Phi$, there is a unique way to write*

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha.$$

*The **height** of β (relative to Δ), denoted $ht(\beta)$, is $ht(\beta) := \sum_{\alpha \in \Delta} c_\alpha$.*

Question 4: In your own words, what is the height measuring?

Question 5: Consider the B_2 root system with

$$\Phi = \{\pm e_1, \pm e_2, \pm(e_1 + e_2), \pm(e_1 - e_2)\} \quad \text{and base } \Delta = \{e_1, -e_1 + e_2\}.$$

(1) Express the remaining roots in Φ in terms of the simple roots in Δ.

(2) Find the height of the following roots with respect to Δ:

(a) e_1, (b) e_2, (c) $e_1 + e_2$, (d) $-e_1 - e_2$

***Question 6:** If β is a positive root but not a simple root, prove that $ht(\beta) > 1$.

***Question 7:** Let $\beta \in V$ be any vector that is expressible (uniquely) as

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$$

for some scalars $c_\alpha \in \mathbb{R}$ where all $c_\alpha \geq 0$ or all $c_\alpha \leq 0$. Prove that either

- β is a scalar multiple of a root in Φ, or
- there exists a Weyl group element $w \in W$ such that

$$w(\beta) = \sum_{\alpha \in \Delta} c'_\alpha \alpha \text{ with some } c'_\alpha > 0 \text{ and some } c'_\alpha < 0.$$

***Question 8:** Show that the Weyl group of B_ℓ is the group of transformations expressible as a composition of a permutation of the entries and an arbitrary number of sign changes. (By a *sign change* we mean the linear transformation that changes the sign of one entry and leaves all others the same.)

Question 9:

(1) Prove that Weyl groups of $A_1 \times A_1$, A_2, B_2, G_2 are dihedral groups of order 4, 6, 8 and 12, respectively.

*(2) Prove that if Φ is any root system of rank 2, its Weyl group must be isomorphic to one of these.

Question 10: Prove that the Weyl group W of a root system Φ is a normal subgroup of $\mathrm{Aut}(\Phi)$, the group of linear automorphisms of V preserving Φ.
Question 11: Determine the Weyl group of A_l.

Question 12: At the end of the lecture, Dr. Emory mentions the Lie algebras that our root systems correspond to. This question is meant to familiarize us with Lie algebras. A *Lie algebra* is a vector space V with a binary operation, called the Lie bracket, denoted $[\cdot,\cdot] : V \times V \to V$ that is bilinear, anti-symmetric, and satisfies the Jacobi identity.[1]
In this problem, we consider the example

$$\mathfrak{sl}(3,\mathbb{C}) := \{A \in M_{3,3}(\mathbb{C}) \mid \mathrm{tr}(A) = 0\}$$

which is an 8-dimensional vector space (there's 9 entries in each matrix and the trace zero condition removes one degree of freedom). The Lie bracket in $\mathfrak{sl}(3,\mathbb{C})$ is defined as

$$[A,B] := AB - BA$$

(1) Compute $\left[\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\right]$.

(2) Witness an example of anti-symmetry by computing

$$\left[\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\right]$$

and comparing with part (1).

(3) Verify that the Jacobi identity holds for

$$X = \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, Y = \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}, Z = \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}.$$

(4) Lie algebras matter because they are the tangent space at the identity of a Lie group. To recover the Lie group $SL(3,\mathbb{C})$, we can apply the matrix exponential map to elements in $\mathfrak{sl}(3,\mathbb{C})$. For any square matrix X, the matrix exponential is defined as

$$\exp(X) := Id + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \dots \qquad (1)$$

Compute $\exp\left(\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\right)$ and find $\det(\exp\left(\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\right))$.

[1] Bilinar means that for $X, Y, Z \in \mathfrak{sl}(3,\mathbb{C})$ and a, b scalars $[aX + bY, Z] = a[X,Z] + b[Y,Z]$ and [Z, aX + bY] = a[Z, X] + b[Z, Y]].
Anti-symmetry means that $[X,Y] = -[Y,X]$.
The Jacobi identity means that $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.