PAWS Root Systems: PROBLEM SET 1

Devjani Basu, Marcella Manivel, Mishty Ray, Ajmain Yamin

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Question 1: Describe the difference between a root system and a basis.

Question 2: Verify that A_ℓ is a root system.

Question 3: Verify that $A_1 \times A_1$, B_2, C_2, G_2 are root systems of rank two.

Question 4: For every $\alpha \in \Phi$ for the root system of type A_2, write down s_α and H_α. Now all of the H_α divide the plane into connected components. Visualize these connected components. Recall the terminology associated with these connected components.

Question 5: Fix a finite dimensional real vector space $V := \mathbb{R}^l$ with the standard Euclidean inner product (dot product). For $\alpha \in V$, let H_α denote the hyperplane or subspace perpendicular to α, i.e.

$$H_\alpha = \{\beta \in V : (\alpha, \beta) = 0\}.$$

Let s_α define a reflection, i.e.

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha$$

1. Verify that $s_\alpha^2 = 1$. [So s_α has order 2 in the group of orthogonal transformations $O(V)$.]
2. Let V' be a subspace of V. If a reflection s_α leaves V' invariant, prove that either $\alpha \in V'$ or else $V' \subset H_\alpha$.
3. Refer to the table in Section 5 (Classification of Root System). Show that the order of $s_\alpha s_\beta$ is 2, 3, 4, 6 when $\theta = \frac{\pi}{2}, \frac{\pi}{3}$ *or* $\frac{2\pi}{3}, \frac{\pi}{4}$ *or* $\frac{3\pi}{4}$, $\frac{\pi}{6}$ *or* $\frac{5\pi}{6}$, respectively. [Note that $s_\alpha s_\beta$ = rotation through 2θ].
4. Show by example that $\alpha - \beta$ may be a root even when $(\alpha, \beta) \leq 0$.

Question 6: Let Φ be a set of vectors in a euclidean space V, satisfying items [1]-[4] of Definition 4.4.

1. Prove that the only possible multiples of $\alpha \in \Phi$ which can be in Φ are $\pm 1/2\alpha$, $\pm\alpha$, $\pm 2\alpha$.
2. Verify that $\{\alpha \in \Phi | 2\alpha \notin \Phi\}$ is a root system.