PAWS Root Systems: PROBLEM SET 0

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Welcome to PAWS! Below are the exercises for Problem Set 0. The questions are loosely in ascending order of difficulty. Feel free to skip around and try whatever exercises would be the most helpful for you. Try as many as you can but don't feel like you need to complete them all! Some that we thought might be more difficult are marked with a star (*).

Question 1: (Review of the symmetric group and permutations) If X is a set, a *permutation* of X is a bijection $\alpha : X \to X$. Two such permutations α, β can be composed to give the permutation $\alpha\beta : X \to X$, which is defined by the rule $\alpha\beta(x) = \alpha(\beta(x))$. Under the operation of composition, the set of all permutations of X forms a group $\mathrm{Sym}(X)$, the *symmetric group* on X. If X is the set $\{1, 2, \ldots, n\}$, we write S_n for $\mathrm{Sym}(X)$.

*(1) Convince yourself that the order of S_n is $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$.

(2) Let σ be the permutation in S_4 given by

$$\sigma(2) = 3, \quad \sigma(3) = 4, \quad \sigma(4) = 2, \text{ and } \quad \sigma(1) = 1.$$

Write down σ in *cycle decomposition* and as a *permutation matrix*. Compute σ^2, σ^3, σ^4, and σ^{-1}. What is the order of σ?

(3) Two cycles $(a_1\ a_2\ \ldots\ a_k)$ and $(b_1\ b_2\ \ldots\ b_l)$ in $\mathrm{Sym}(X)$ are *disjoint* if no element of X is moved by both cycles. If $k \geq 2$ and $l \geq 2$, this can be expressed by saying

$$\{a_1, a_2, \ldots, a_k\} \cap \{b_1, b_2, \ldots, b_k\} = \varnothing$$

Write down the (dijoint) cycle decomposition of the permutation α in S_8 given by

$$\alpha(1) = 3, \alpha(2) = 5, \alpha(3) = 7, \alpha(4) = 4, \alpha(5) = 2, \alpha(6) = 8, \alpha(7) = 1, \alpha(8) = 6.$$

(4) Verify that $(1\ 2\ 3\ 4)(2\ 3\ 4) \neq (2\ 3\ 4)(1\ 2\ 3\ 4)$.

(5) A *transposition* is a 2-cycle $(a\ b)$. Any cycle in S_n can be written as a product of transpositions. Try writing the cycle (12345) as a product of transpositons.

Question 2: (Review of the dot product) The *dot product* between two vectors $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{R}^n$ is defined to be

$$v \cdot w := v_1 w_1 + \cdots v_n w_n \in \mathbb{R}.$$

Sometimes we denote the dot product $v \cdot w$ by the alternative notation (v, w). Recall that the dot product tells us about angles between vectors. Specifically,

$$(v, w) = ||v|| \cdot ||w|| \cdot \cos(\theta),$$

where θ is the angle between the vectors. This can be taken to be a <u>definition</u> of angle in higher dimensions.

(1) What is the dot product of $(1, 5, 7)$ and $(4, 2, 2)$?

*(2) Prove that the dot product is a *symmetric bilinear form* on $\mathbb{R}^n$. That is, prove

$$(v, w) = (w, v), \quad (v + w, u) = (v, u) + (w, u), \quad (\lambda v, w) = \lambda(v, w)$$

for all $v, w, u \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. We use the notation (v, w) in place of $v \cdot w$ to emphasize that it is a symmetric bilinear form. Sometimes, we use the notation (v, w) to denote an arbitrary symmetric bilinear form on a vector space V, not just the standard dot product on $\mathbb{R}^n$.

(3) With $v = (1, 1, 1)$ and $w = (2, 4, 6)$, find the dot product (v, w).

(4) Find the lengths of v and w and use them to solve for the angle between v and w.

(5) Use the dot product to find the plane of all vectors perpendicular to v.

Question 3: (All about hyperplanes) If $v \in \mathbb{R}^n$ is a nonzero vector, the notation H_v denotes the *hyperplane* perpendicular to v, i.e.

$$H_v = \{w \in \mathbb{R}^n \mid (v, w) = 0\}.$$

Observe that when $n = 2$, H_v is a line in $\mathbb{R}^2$, and when $n = 3$, H_v is a plane in $\mathbb{R}^3$. In general, H_v is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, or a hyperplane passing through the origin $(0, \dots, 0)$.

(1) In $\mathbb{R}^2$, draw the line H_v for $v = (1, 2)$. Observe that reflection in the line H_v is the linear transformation $s_v : \mathbb{R}^2 \to \mathbb{R}^2$ given by the formula

$$s_v(w) = w - \frac{2(w, v)}{(v, v)} v.$$

(2) In $\mathbb{R}^3$, draw the plane H_v with $v = (2, 1, 0)$. Observe that reflection in the plane H_v is the linear transformation $s_v : \mathbb{R}^3 \to \mathbb{R}^3$ given by the formula

$$s_v(w) = w - \frac{2(w, v)}{(v, v)} v.$$

*(3) In general, prove that for any nonzero $v \in \mathbb{R}^n$, reflection in the hyperplane H_v is the linear transformation $s_v : \mathbb{R}^n \to \mathbb{R}^n$ given by the formula

$$s_v(w) = w - \frac{2(w, v)}{(v, v)} v.$$

That is, prove $s_v(v) = -v$ and $s_v(w) = w$ for all $w \in H_v$.

Question 4: Consider $\mathbb{R}^2$ with the inner product given by dot product and the standard basis $e_1 = (1, 0), e_2 = (0, 1)$. Set

$$\Phi = \{e_1 - e_2, e_2 - e_1\}.$$

(1) What is the span of Φ?

(2) For $\alpha \in \Phi$, let s_α denote the reflection through the hyperplane H_α perpendicular to α as in the previous question. A set S is *preserved* by a linear transformation T if $T(s) \in S$ for all $s \in S$. Is Φ preserved by s_α?

(3) Set $\langle \lambda, \mu \rangle := \dfrac{2(\lambda, \mu)}{(\mu, \mu)}$. Compute $\langle e_1 - e_2, e_2 - e_1 \rangle$. What do you observe?

Question 5: Consider $\mathbb{R}^3$ with the inner product given by dot product and the standard basis $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$. Set

$$\Phi = \{e_1 - e_2, e_2 - e_1, e_1 - e_3, e_3 - e_1, e_2 - e_3, e_3 - e_2\}.$$

(1) What is the span of Φ?

(2) Is Φ preserved by the hyperplane reflections s_α for $\alpha \in \Phi$?

(3) Set $\langle \lambda, \mu \rangle := \dfrac{2(\lambda, \mu)}{(\mu, \mu)}$. Compute $\langle e_1 - e_2, e_2 - e_3 \rangle$, $\langle e_1 - e_3, e_3 - e_2 \rangle$ and $\langle e_1 - e_2, e_1 - e_3 \rangle$. What do you observe?

Question 6: We have an *action* of S_n on $V_n = \mathbb{R}^n$ given by permuting the n coordinates. In particular, if $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ is a vector and $\sigma \in S_n$ is a permutation in S_n, then the action of S_n on V may be written explicitly as

$$\sigma \cdot v = \sigma \cdot (v_1, v_2, \ldots, v_n) = (v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, \ldots, v_{\sigma^{-1}(n)}). \quad (1)$$

(1) Write down the explicit 3×3 *permutation matrices* coming from the action of S_3 on $\mathbb{R}^3$.

(2) Consider $V_4 = \mathbb{R}^4$ with the standard basis $\{e_1, e_2, e_3, e_4\}$. Let a permutation in S_4 act on V_4 by permuting the standard basis vectors, as described above. Now consider the transpositions (1 3), (2 4), and (2 3). Where do these transpositions send the following vectors?

- $e_1 - e_2$
- $e_1 - e_3$
- $e_2 - e_4$
- $-e_2 + e_3$

Comment on how the transposition $(i\ j)$ acts on $e_i - e_j$.

*(3) Prove that the action of S_n on $\mathbb{R}^n$ described above preserves the dot product on $\mathbb{R}^n$. That is, prove

$$(v, w) = (\sigma \cdot v, \sigma \cdot w)$$

for all $v, w \in \mathbb{R}^n$ and all $\sigma \in S_n$.

Question 7: Write V_n to denote $V_n = \mathbb{R}^n$ equipped with the natural action of S_n given by permuting coordinates as defined in Equation (1). A subspace W of V_n is *preserved* by S_n if $\sigma \cdot w \in W$ for all $\sigma \in S_n$ and $w \in W$. A subspace W of V_n is *fixed pointwise* by S_n if $\sigma \cdot w = w$ for all $\sigma \in S_n$ and $w \in W$.

(1) Consider $V_2 = \mathbb{R}^2$ and the two subspaces of V_2 spanned by $e_1 - e_2$ and $e_1 + e_2$. Which of these subspaces is preserved by the group S_2? Which of these subspaces is fixed pointwise under the action of S_2?

(2) What is the orthogonal complement of the subspace $\text{span}\{e_1 - e_2\}$ in V_2? Here, V_2 is equipped with the usual dot product.

*(3) Find a subspace of V_3 which is fixed pointwise by the action of S_3. Justify your argument. Can you generalize your argument to V_n with the action of S_n?

Question 8: (*) The *orthogonal group* $\mathrm{O}(n, \mathbb{R})$ is set of orthogonal matrices, i.e. the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of matricies $A \in \mathrm{GL}_n(\mathbb{R})$ which preserve the dot product, i.e.

$$(v, w) = (Av, Aw)$$

for all $v, w \in \mathbb{R}^n$. Convince yourself that S_n is a subgroup of $\mathrm{O}(n, \mathbb{R})$, then prove that the transpositions are the sole reflections belonging to S_n.