PAWS 2024: SYMMETRIES OF ROOT SYSTEMS

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1. Introduction

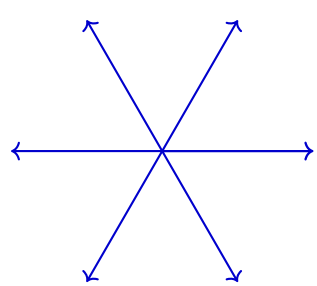
These lecture notes are in draft form and are created for the Preliminary Arizona Winter School 2024: Symmetries of root systems course. The hope for these lecture notes is to make the subject accessible to advanced undergraduate and early graduate students most of whom have a semester in Abstract Algebra. The main sources for the notes are [Hum90], [Car89], [Hal15].

2. Intuitive Definition

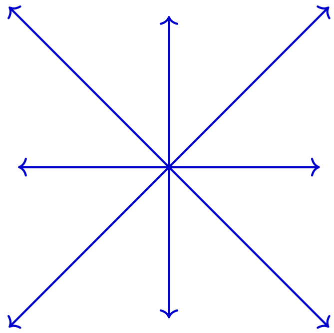
A root system is a "very symmetrical" set of vectors in n-dimensional Euclidean space. The classical motivation for studying root systems is their role in the classification of semi-simple Lie algebras, i.e. classical and exceptional groups, over the complex numbers. The root datum has connections to the Langlands dual group as well as L-functions. Thus, root systems have connections to representation theory, number theory, algebra, geometry, and physics.

Each of the following examples are in $\mathbb{R}^2$. How would one describe the symmetries that you see in the pictures? What properties can you derive?

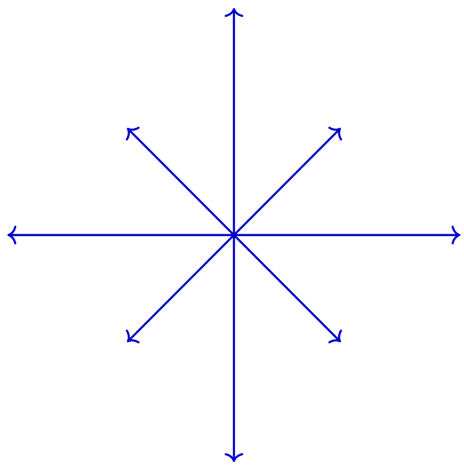
Type A_2



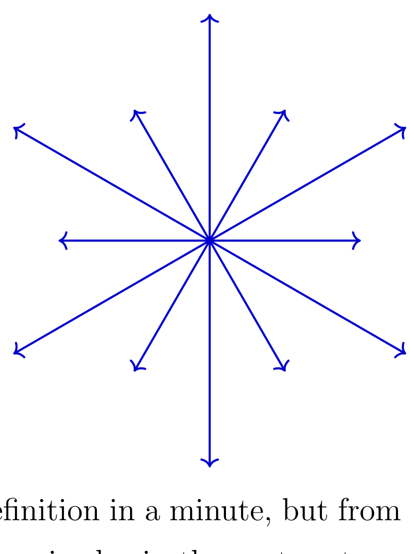
Type B_2



Type C_2



Type G_2



We will do an actual definition in a minute, but from these pictures, we see that

(1) For any vector v, $-v$ is also in the root system

(2) If you take any vector v and look at the in the perpendicular to v, reflect he whole picture across that line, the reflection corresponds to the original picture.

Let V be a real Euclidean space endowed with a positive definite symmetric bilinear form (λ, μ). A reflection s is a linear operator on V on which it sends a nonzero vector α to $-\alpha$ and fixes pointwise the hyperplane H_α orthogonal to α. We typically write $s = s_\alpha$ understanding that for any $c \in \mathbb{R}, s_{c\alpha} = s_\alpha$. A finite subgroup of reflections is an interesting type of finite subgroup of $\mathrm{O}(V)$ and there is such a classification of all such groups!

Note that

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \tag{2.1}$$

Once can easily compute that $s_\alpha^2 = 1$ and so s_α has order 2 in the group of orthogonal transformations $\mathrm{O}(V)$.

3. Basic Examples of Types of Root Systems

In our intuitive description of root systems, we only dealt with Rank 2. But these generalize to higher rank in a natural way.

3.1. **($A_{n-1}, n \geqslant 2$).** Consider the symmetric group $\mathcal{S}_n$. $\mathcal{S}_n$ can be though of as a subgroup of $\mathrm{O}(n, \mathbb{R})$. We can see this by noting that $\mathbb{R}^n$ has a standard basis of vectors $\{e_1, e_2, \ldots, e_n\}$. We can make a permutation act on V by permuting the standard basis vectors (permuting the subscripts). Recall that a transposition is a permutation which exchanges two elements and keeps all other fixed. Moreover, every permutation can be written as a product of transpositions. So what happens with the transposition (ij)? Let's consider $\mathbb{R}^4$. We can see that the trasnposition (23) as as a reflection sending $e_2 - e_3$ to it negative and fixing pointwise the orthogonal compliment. Not that this is all the vectors having equal ith and jth component). As we stated before and was in Problem set 0,, $\mathcal{S}_n$ is generated by transpositions so it is a reflection group. Indeed, $\mathcal{S}_n$ is generated by the transpositions $(i, i+1), 1 \leqslant i \leqslant n-1$.

3.2. **($B_n, n \geqslant 2$).** Let V be $\mathbb{R}^n$, so $\mathcal{S}_n$ acts on V by permuting the basis vectors as before. There are other reflections sending e_i to $-e_i$ and fixing the other e_j. Since the dimension of V is n, these sign changes generate of group of order 2^n isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. This group of sign changes is

- normalized by $\mathcal{S}_n$
- intersects $\mathcal{S}_n$ trivially.

Conjugating the sign change e_1 to $-e_1$ by a transpositions gives another sign change so the semidirect product with $\mathcal{S}_n$ is a group of order $2^n n!$ and is a reflection group that we call W. In general this will normally be a Weyl group which we will discuss later. The Weyl group can be thought of as the skeleton of the corresponding group of Lie type.

3.3. **($D_n, n \geqslant 4$).** There is another reflection group acting on $\mathbb{R}^n$ by permuting the standard basis vectors, it is a subgroup of index 2 in the group of type B_n above. Note that $\mathcal{S}_n$ normalizes the subgroup consisting of sign changes which involve an even number of signs, generated by the reflections sending $e_i + e_j$ to $-(e_i + e_j), i \neq j$. So the semidirect product is also a reflection group.

4. Root Systems

The origins of why these are called roots, come from the roots of characteristic polynomials. One of the reasons that we study root systems is that many groups have associated root data and this root data determines this group, up to isomorphism. Around the time period of 1880-1900 Cartan and Killing proved the following.

Theorem 4.1. *Every semisimple Lie algebra over the complex numbers has an associated root system and the root system determines the Lie algebra (up to isomorphism).*

The result was generalized by Chevalley in the 1940s and 1950s for reductive groups. That is, he showed that

Theorem 4.2. *Every reductive algebraic group has associated root data, and up to isomorphism, this root data determines the group.*

Next we define roots systems more formally. We fix a finite dimensional real vector space $V := \mathbb{R}^l$ with the standard Euclidean inner product, which is also known as the dot product.

Definition 4.3. *For $\alpha \in V, H_\alpha$ denotes the hyperplane or subspace perpendicular to α, i.e.*

$$H_\alpha = \{\beta \in V : (\alpha, \beta) = 0\}.$$

Definition 4.4. *A subset Φ of V is called a root system of V if the following axioms are satisfied*

(1) *Φ is a finite set of non-zero vectors*
(2) *Φ spans V.*
(3) *if $r, \lambda \in \Phi$, then*

$$s_r(\lambda) = \lambda - \frac{2(r,\lambda)}{(r,r)} r \in \Phi.$$

Another way to say this is that every root is closed under reflection through the hyperplane perpendicular to r

(4) *(Integrability)If $r, \lambda \in \Phi$ then $\langle \lambda, r\rangle = \dfrac{2(r,\lambda)}{(r,r)}$ is an integer.*

An element $\alpha \in \Phi$ is a ***root****.*

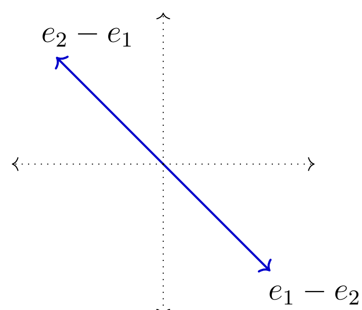
We will also assume in the definition for our root system, that it is ***reduced****, that is to say that the only scalar multiples of a root $\lambda \in \Phi$ are λ and $-\lambda$.*

Not all definitions have item (4) - this is also called a Crystallographic root system. Since Φ spans V, the dimension of V over the real numbers is l and is an invariant of Φ called the **rank** of Φ, and denote it by $rk(\Phi)$.

Definition 4.5. *Let Φ be a root system in V. For $\alpha \in \Phi$, the hyperplanes H_α divide V into connected components. These connected components of $V \setminus \bigcup_\alpha H_\alpha$ are the* ***Weyl Chambers*** *of Φ (We will see pictures in a minute, and will make sense then.)*

4.1. **Examples.** We will work in $\mathbb{R}^l$ with standard basis $e_1, e_2, \dots e_l$ with the standard Euclidean inner product, also known as the dot product.

Example 4.6. *The type A_1 root system. For this example we will work in $\mathbb{R}^2$ with the standard basis e_1, e_2. Let $\Phi = \{e_1 - e_2, e_2 - e_1\}$. Geometrically we can represent this with the following picture*

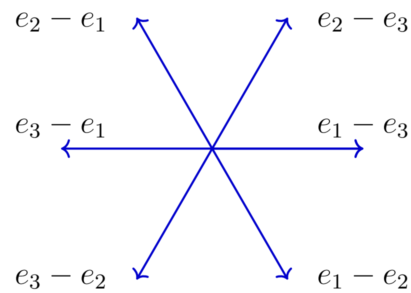


Let V be the span of $(-1, 1)$. Then Φ is a root system in E. Let's check the integrability property (iv) in Def 4.4.

$$\frac{2(e_1 - e_2, e_2 - e_1)}{(e_2 - e_1, e_2 - e_1)} = \frac{2(-1-1)}{(1+1)} = -2.$$

This is the root system of type A_1 which corresponds to the dimension of V, but we drew this in $\mathbb{R}^2$ to generalize this to the rank 2 examples below.

Example 4.7. *The type A_2 root system. Consider $\mathbb{R}^3$ with the standard basis vectors e_1, e_2, e_3, Let $\Phi = \{e_1 - e_2, e_2 - e_1, e_1 - e_3, e_3 - e_1, e_2 - e_3, e_3 - e_2\}$. The span of Φ is the plane with normal vector $e_1 + e_2 + e_3$. Let V be this subspace. We claim that Φ is a root system in V. Geometrically, we can represent this with the following picture.*



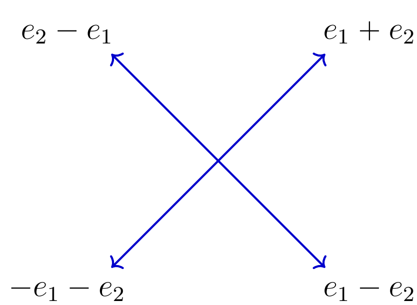
To check integrability, one must check all the cases such as

$$\frac{2(e_1 - e_2, e_2 - e_3)}{(e_1 - e_2, e_1 - 2_2)} = \frac{2(-1)}{2} = -1.$$

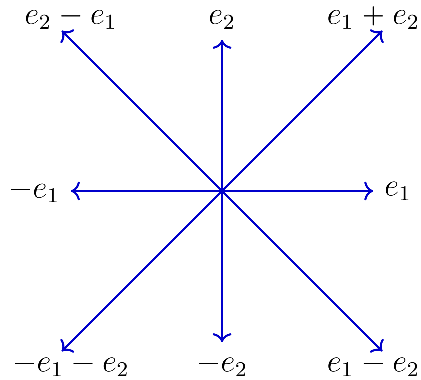
Bot notice there are a lot of cases to check!

Example 4.8. *The A_l root system. Let $e_1, e_2, \ldots, e_{l+1}$ be the standard basis of $\mathbb{R}^{l+1}$. Let $\Phi = \{\pm(e_i - e_j) : 1 \leqslant i|j \leqslant l+1\}$. Let $V \subset \mathbb{R}^{l+1}$ be the span of Φ with the inner form being the dot product.*

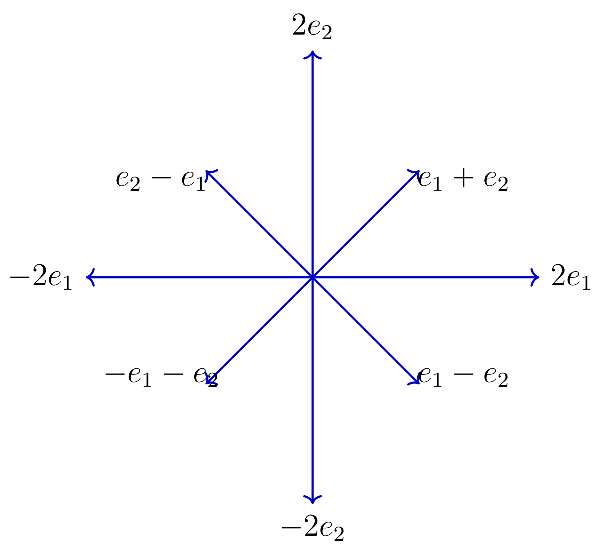
Example 4.9. *$A_1 \times A_1$ root system. Let e_1, e_2 be the standard basis. Consider $\mathbb{R}^2$ with the dot product. We have two copies of the A_1 root system. One given as before with $\{e_1 - e_2, e_2 - e_1\}$ and the other given by $\{e_1 + e_2, -e_1 - e_2\}$. Let $\Phi = \{e_1 - e_2, e_2 - e_1, e_1 + e_2, -e_1 - e_2\}$ and as before we can represent geometrically in the following picture.*



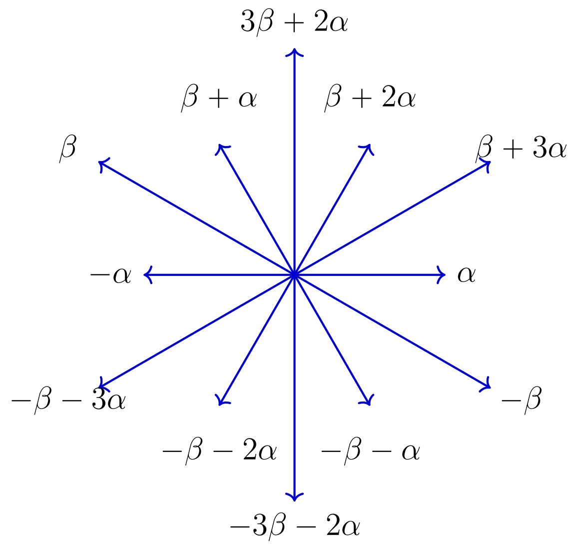
Example 4.10. *The B_2 root system. Consider $\mathbb{R}^2$ with the dot product with basis e_1, e_2. Let $\Phi = \{\pm e_1, \pm e_2, \pm(e_1 + e_2), \pm(e_1 - e_2)\}$. Then we can show that this is a root system, and can be represented geometrically as follows: Type B_2*



Example 4.11. *The Type C_2 root system. In $\mathbb{R}^2$ with the dot product and* $\Phi = \{\pm 2e_1, \pm 2e_2, \pm(e_1 + e_2), \pm(e_1 - e_2)\}$



Example 4.12. *The Type G_2 root system. Now consider $\mathbb{R}^3$ with basis elements e_1, e_2, e_3. Let $\Phi = \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3), \pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\}$. Note that the fist six vectors are the same as for A_2 and these are in the hyperplane perpendicular to $e_1 + e_2 + e_3$. The other vectors are in the same plane, so let V be the plane. Let $\alpha = e_1 - e_2$ and $\beta = 2e_2 - e_1 - e_3$.*



As it turns out, we have now described all of the irreducible root systems of rank 2.

This is a nice place to give the following definitions that will be needed for Problem set 2. Let Φ be a root system. The set of **positive roots**, which we denote by Φ^+ is a subset of Φ such that

- For each $\alpha \in \Phi$ exactly one of α or $-\alpha$ is contained in Φ^+.
- For any two distince roots $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta$ is a root, $\alpha + \beta \in \Phi^+$.

An element in Φ^+ is called a **simple root** if it can not be written as the sum of two elements in Φ^+. The set of simple roots is referred to as a base for Φ.

5. Classification of Root Systems

The integrability condition in Def 4.4 restricts what angles are possible. We define the symbol,

$$\langle \beta, \alpha \rangle := 2\frac{(\alpha, \beta)}{(\beta, \beta)}.$$

Since $\langle \beta, \alpha \rangle$ and $\langle \alpha, \beta \rangle$ are integers we have that

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} \cdot 2\frac{(\alpha, \beta)}{(\beta, \beta)}$$

$$= 4\frac{(\alpha, \beta)^2}{|\alpha|^2 |\beta|^2}$$

$$= 4\cos^2\theta = (2\cos\theta)^2.$$

Since $2\cos\theta \in [-2, 2]$ and $(2\cos\theta)^2$ is an integer,

$$\cos\theta \in \left\{0, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}, \pm 1\right\}$$

which corresponds to the angles

$$\left\{\frac{\pi}{2}, \frac{\pi}{3} \text{ or } \frac{5\pi}{3}, \frac{\pi}{4} \text{ or } \frac{7\pi}{4}, \frac{\pi}{6} \text{ or } \frac{11\pi}{6}, 0 \text{ or } \pi\right\}.$$

We are wanting a reduced system, so no scalar multiples of a root α other than α and $-\alpha$ so 0 and π are not included, these correspond to 2α and -2α. Another way to look at this is if $4\cos\theta = 4$ and then $\theta = \pi$ which would make $\beta = \pm\alpha$. The gives the following possibilities:

$\langle\alpha,\beta\rangle$	$\langle\beta,\alpha\rangle$	θ	$\left(\frac{\lvert\beta\rvert}{\lvert\alpha\rvert}\right)^2$
0	0	$\frac{\pi}{2}$	unrestricted
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

In the A_l root system, all roots have equal length. In the B_2 and C_2 root systems, two roots have length corresponding to 2 (the square of the ratio of the lengths). In the G_2 root system, two roots have length corresponding to 3 and all the angles are multiples of $\frac{\pi}{6}$. This ends up leading to a classification of root systems.

We begin with a the following proposition which can be found in [Hal15], and is also left as an exercise in Problem Set 3.

Proposition 5.1. *Suppose α and β are roots, α is not a multiple of β and $\langle\alpha, \alpha\rangle \geqslant \langle\beta, \beta\rangle$. Then one of the following holds.*

(1) $\langle\alpha, \beta\rangle = 0$
(2) $\langle\alpha, \alpha\rangle = \langle\beta, \beta\rangle$ *and the angle between α and β is $\pi/3$ or $2\pi/3$.*
(3) $\langle\alpha, \alpha\rangle = 2\langle\beta, \beta\rangle$ *and the angle between α and β is $\pi/4$ or $3\pi/4$.*
(4) $\langle\alpha, \alpha\rangle = 3\langle\beta, \beta\rangle$ *and the angle between α and β is $\pi/6$ or $5\pi/6$.*

Proposition 5.2. *Every rank two root system is isomorphic to $A_1 \times A_1, A_2, B_2$, or G_2.*

Proof. We assume that $V = \mathbb{R}^2$; and let $\Phi \subset \mathbb{R}^2$ be a root system and let θ be the smallest angle occurring between any two vectors in Φ. Since the element in Φ spans $\mathbb{R}^2$, we can find two linearly independent vectors α and β in Φ. If the angle between α and β is greater than $\pi/2$, then the angle between α and $-\beta$ is less than $\pi/2$; thus the minimum angle is at most $\pi/2$. From 5.1, $\theta \in \{\pi/2, \pi/3, \pi/4, \pi/6\}$. Let α and β be two elements of Φ such that the angle between them is the minimum angle θ. Then the vector $-s + \beta(\alpha)$ will be a vector that is at angle θ to β but on the opposite side of β from α. Thus, $-s_\beta(\alpha)$ is at angle 2θ to α. Then $-s_{s_\beta \cdot \alpha}\beta$ is at most 3θ to α. Continuing we get vectors at angle $n\theta$ to α for all n. Since a nontrivial positive multiple of a roots is not a root, these vectors are unique. Each of the allowed values of θ eveny divides 2π, we will eventually get to α again (else there would be an angle smaller than θ).

Thus ϕ must consist of n equally spaced vectors with consecutive vectors separated by angle θ, where θ is one of the acute angles in Proposition 5.1. If say, $\theta = \pi/4$ then in order to satisfy the length requires of Proposition 5.1, the roots must alternate between shorter length and a second length that is greater by a factor of $\sqrt{2}$. Thus our root system must be isomorphic to B_2. Similar reasoning shows that all remaining values of θ yield one of the root systems $A_1 \times A_1, A_2, B_2, G_2$. $\square$

6. A base of a root system and Weyl Chambers

Definition 6.1. *A nonempty root system Φ us* ***irreducible*** *if it is not the direct sum of two nonempty root systems.*

Definition 6.2. *A nonempty root system Φ is* ***reducible*** *if it can be written as a disjoint union of nonempty root systems, i.e.*

$$\Phi = \Phi_1 \bigsqcup \Phi_2$$

where Φ_1 and Φ_2 are root systems.

Every root system is the direct sum of some set of irreducible root systems, and this composition is unique up to the order of the terms. This is why for classification purposes, we only need to consider irreducible root systems.

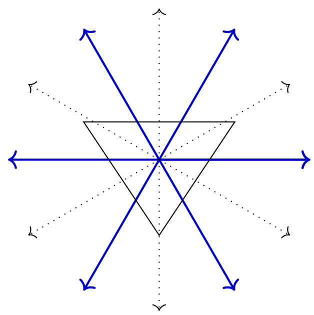
Definition 6.3. *Two root systems (V_1, Φ_1) and (V_2, Φ_2) are* ***isomorphic*** *if there is an invertible linear transformation V_1 to V_2 that maps Φ_1 to Φ_2 such that for each pair of roots, the number $\langle x, y\rangle$ is preserved.*

6.1. Weyl group.

The group W generated by all reflections s_α, where $\alpha \in \Phi$, is known as the Weyl group of Φ.

Recall that s_α is the reflection through the hyperplane of α. Since Φ is a root system, each s_α preserves Φ so $W(\Phi)$ may also be viewed as a subgroup of the permutation group of Φ. Note that $W(\Phi)$ is then finite.

Example 6.4. *The Weyl group of A_2. Recall that the root system of A_2 is*



The Weyl group of this system is the subgroup of the symmetry group of A_2 generated by reflections. This is the symmetry group of an equilateral triangle, which is $\mathcal{S}_3$. (Think of as the vertices of an equilateral triangle).

Note that W is not the full symmetry group of the root system. If we rotate by 60 degrees, Φ is preserved, but not as an element of W.

6.2. **Weyl Chambers.** If $\Phi \subset V$ is a root system, consider the hyperplane H_α perpendicular to each root α and recall that s_α denotes the reflection about the hyperplane. From before, the Weyl group is the group of reflections generated by the $s'_\alpha s$. Note that the complement of this set of hyperplanes is disconnected and each component is called a **Weyl chamber**.

If Φ is a root system, a subset Δ of Φ is called a base if the following conditions hold"

(1) Δ is a basis for V as a vector space.
(2) Each root α can be expressed as a linear combination of elements of Δ with linear coefficients in such a way that these coefficients are all non-positive or all non-negative.

The non-negative roots are called **positive roots** and the non-positive roots are called **negative roots**. The elements of Δ are called **simple roots**.

Fix a set Δ of simple roots, the fundamental Weyl chamber associated to Δ are the set of points $v \in V$ such that $(\alpha, v) > 0$ for all $\alpha \in \Delta$.

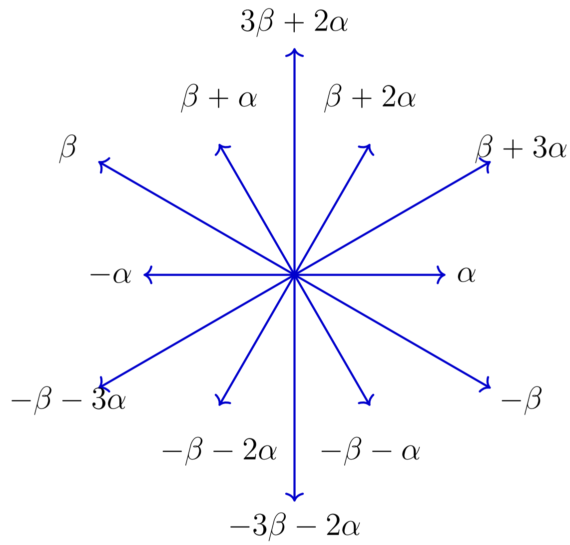
Example 6.5. *Recall that for A_2,*

$$\Phi = \{e_1 - e_2, e_2 - e_1, e_1 - e_3, e_3 - e_1, e_2 - e_3, e_3 - e_2\}.$$

And as it turns out,

$$\Delta = \{e_1 - e_2, e_3 - e_1\}.$$

Example 6.6. *Base for G_2. Recall from, before that we had $\alpha = e_1 - e_2$ and $\beta = 2e_2 - e_1 - e_3$ and the roots were linear combinations of α and β. Then $\Delta = \{\alpha, \beta\}$ for G_2. Which makes sense when we look at the geometric interpresation of the root system for G_2.*



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Theorem 6.7. *The order of the Weyl group is equal to the number of Weyl chambers.*

Proof. See Hall, Proposition 8.23 and Proposition 8.27. □

Recall from our A_2 example, the Weyl group has 6 elements and there are 6 Weyl chambers.

7. Cartan Matrices and Dynkin Diagrams

Theorem 7.1. *(Existence Theorem) Let Φ be an irreducible root system. Then there exists a simple Lie algebra over $\mathbb{C}$ which has a root system equivalent to Φ.*

A proof of the existence theorem can be found in [Tit66].

Theorem 7.2. *(Isomorphism theorem) Any two simple Lie algebras over $\mathbb{C}$ with equivalent root systems are isomorphic.*

A proof of this theorem is in Jacobson[1]. This can be deceptive though.

Let Φ be a root system with base $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Recall that $\langle \alpha, \beta \rangle = \dfrac{2(\alpha, \beta)}{(\beta, \beta)}$. The isomorphism theorem gives a matrix in [Car89, pg. 43] called the **Cartan** matrix given by

$$(A_{ij}) = (\langle \alpha_i, \alpha_j \rangle)$$

which is an $l \times l$ matrix with entries in the integers thanks to property (4) of Definition 4.4 and this does not depend on the base that we choose.

Remark 7.3. *Note that the diagonal elements of the Cartan matrix are $\langle \alpha_i, \alpha_i \rangle = 2$ for all i.*

Example 7.4. *Cartan matrix for G_2. We use the base for G_2 of $\alpha = e_1 - e_2$ and $\beta = 2e_2 - e_1 - e_3$ for G_2. Then since we already know that the diagonal elements are 2 from Remark 7.3 we just need to compute the following*

$$\langle \alpha, \beta \rangle = \frac{2(e_1 - e_2, 2e_2 - e_1 - e_3)}{2e_2 - e_1 - e_3, 2e_2 - e_1 - e_3)} = \frac{2(-1-2)}{1+4+1} = -1$$
$$\langle \beta, \alpha \rangle = \frac{2(2e_2 - e_1 - e_3, e_1 - e_2)}{(e_1 - e_2, e_1 - e_2)} = \frac{2(-1-2)}{2} = -3$$

Thus the Cartan matrix for G_2 is

$$G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

The Cartan matrices of the other individual simple algebras are shown below:

$$A_l : \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & 0 & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & & & & & & \\ & & & & & & & & \\ & & & & & & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & 0 & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \end{pmatrix},$$

$$B_l : \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & 0 & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & & & & & & \\ & & & & & & & & \\ & & & & & & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & 0 & & & & -1 & 2 & -1 \\ & & & & & & & -2 & 2 \end{pmatrix},$$

$$C_l : \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & 0 & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & & & & & \\ & & & & & & & \\ & & & & & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & 0 & & & -1 & 2 & -2 \\ & & & & & & -1 & 2 \end{pmatrix},$$

$$D_l : \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & 0 & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & & & & & \\ & & & & & & & \\ & & & & & -1 & & \\ & & & & -1 & 2 & -1 & -1 \\ & & 0 & & & -1 & 2 & 0 \\ & & & & & -1 & 0 & 2 \end{pmatrix},$$

$$F_4 : \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

$$E_6 : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$E_7 : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$E_8 : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

In the last lecture we will see an example of using these matrices!

8. Dynkin Diagrams

Let Φ be a root system with base Δ, the associated Dynkin diagram corresponds to the roots in Δ. Edges are drawn between the vertices as follows:

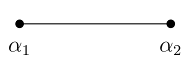
- If the vectors are orthogonal, there is no edge.
- If the vectors form an angle of 120 degrees, there is an undirected single edge
- If the vectors form an angle of 135 degrees, there is a directed double edge
- If the vectors form an angle of 150 degrees, there is a directed triple edge.

We can also form the Dynkin Diagrams by reading off the Cartan matrices. The vertices are lements of Δ and between the two vertices α, β the number of edges is

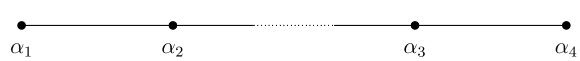
$$\#Edges(\alpha, \beta) = \max\left(|\langle \alpha, \beta \rangle, |\langle \beta, \alpha \rangle|\right).$$

If one of the roots is a longer root, then we direct the multiple edges pointing toward the longer root.

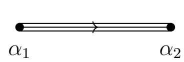
Example 8.1. *Dynkin diagram for type A_2. There are only two roots in the base of A_2 so the Dynkin diagram has two vertices , call them α_1, α_2. Between them is just one edge and they have the same length so the graph is not directed. So the Dynkin diagram for A_2 is*



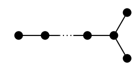
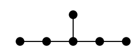
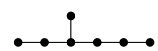
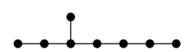
Example 8.2. *Dynkin diagram for type A_l. We use the same base as before for tyoe A_l and use the Cartan matrix to read off the edges. Note that all of the edges have the same length.*



Example 8.3. *Dynkin diagram for type G_2. We have the simple roots, we will call them α_1, α_2 with α_2 being the longer root. From the Cartan matrix we have the $|\langle \alpha_2, \alpha_1 \rangle| = 3$, so there are three edges between the two vertices. We also have an arrow pointing to the longer root. Hence, we have*



Follows is the description of the simple Lie algebras over the complex numbers. For each algebra, we have the dimension, the rank, the number N of positive roots, the order of the Weyl group W and the Dynkin diagram.

Type	Dim	Rank	N	—W—	Dynkin diagram
$A_l(l \geqslant 1)$	$l(l+2)$	l	$\frac{1}{2}l(l+1)$	$(l+1)!$	
$B_l(l \geqslant 2)$	$l(2l+1)$	l	l^2	$2^l \cdot l!$	
$C_l(l \geqslant 3)$	$l)2l+1)$	l	l^2	$2^l \cdot l!$	
$D_l(l \geqslant 4)$	l(2l-1)	l	l^2	$2^{l-1} \cdot l!$	
G_2	14	2	6	12	
F_4	52	4	24	$2^7 \cdot 3^2$	
E_6	78	6	36	$2^7 \cdot 3^4 \cdot 5$	
E_7	133	7	63	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	
E_8	248	8	120	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	

9. Root systems and the Langlands program

This lecture is coming to these notes soon!

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