Asymptotics of reductive p-adic groups

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What are the "asymptotics" ?

Why are the "asymptotics" interesting ?

Let F be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, $\underline{G}$ a connected reductive group over F, $G = \underline{G}(F)$, R a field, π an irreducible smooth representation of G on an R-vector space V.

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"asymptotics" , "around the identity" , "more uniform" this is vague !

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"around the identity" means "on $G_{d(\pi)+} = \cup_{x \in \mathcal{B}(G)} G_{x,d(\pi)+}$"

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Hence

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This is what we mean by "more uniform" for a finite dimensional smooth R-representation !

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When $\dim \pi$ is infinite what means "more uniform" ?

That's going to be the theme of my lecture !

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$$\pi = \sum_\lambda c_\pi(\lambda)\,\mathrm{ind}_{P_\lambda}^G 1, \quad c_\pi(\lambda) \in \mathbb{Z}, \quad \text{on } G_{d(\pi)+}.$$

We proved only $\sim$ but we prove it when p is large enough.

Proof: Reduction to $R = \mathbb{C}$ using that any cuspidal irrreducible representation of $GL_n(D)$ over $\mathbb{F}_\ell^{ac}$ lifts to $\mathbb{Q}_\ell^{ac}$ (Minguez-Sécherre).

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0 iff $\dim V < \infty$ iff $\lambda = n$.
$d(n-1)$ iff $\lambda = (n-1, 1)$
$dn(n-1)/2$ iff (π, V) is generic iff $\lambda = (1, \dots, 1)$.

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$$G_{0+} = I_{1/2},\ G_{(1/2)+} = K_1,\ G_{1+} = I_{3/2},\ G_{(3/2)+} = K_2, \dots$$

$$G_{d(\pi)+} = \begin{cases} I_{d(\pi)+1/2} & \text{if } d(\pi) \text{ is an integer} \\ K_{d(\pi)+1/2} & \text{if } d(\pi) \text{ is a half-integer} \end{cases}$$

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$a_\pi = \dim JL(\pi)$ for a supercuspidal representation
$JL(\pi)$ the R-representation of D^* of dimension >1 image of π by Jacquet-Langlands, D the quaternion division F-algebra.

$$\dim V^{I_i} = \dim V^{I_{i+1/2}} = -a_\pi + 2q^i, \quad \dim V^{K_i} = -a_\pi + (q+1)q^i.$$

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For $GL_n(D)$ there were two main steps in the proof:

- $R = \mathbb{C}$ harmonic analysis
- Reduction modulo ℓ, prime number $\ell \neq p$.

Harmonic analysis ($R = \mathbb{C}, \mathrm{char}(F) = 0$, any G)

(Harish-Chandra)

• The distribution character of π is represented by a

(*) locally integrable G-invariant function $\theta_\pi(g)$ on G, locally constant on G_{reg}

$$\mathrm{tr}(\pi(f)) = \int_G f(g)\theta_\pi(g)\,dg, \quad f \in C_c^\infty(G).$$

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• One can choose an O_F-lattice $\mathcal{L}$ in $\mathrm{Lie}(G)$ on which the exponential map is defined and such that $K = \exp(\mathcal{L})$ is a group.

• (Harish-Chandra) The Fourier transforms $\hat{\mu}_{\mathcal{O}}$ of nilpotent orbital integrals in $\mathrm{Lie}(G)$ give a finite basis $I(G)$ of the space of locally integrable G-invariant functions of G, locally constant on G_{reg} on $\cup_{g \in G} gKg^{-1}$

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Local character expansion

$\mathrm{tr}(\pi) \sim \sum_{i\in I(G)} c_\pi(i)\, i$

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• If the nilpotent orbit giving i has maximal dimension among the nilpotents orbits such that $c_\pi(i) \neq 0$, then $c_\pi(i)$ is the dimension of a space of (possibly degenerate) Whittaker functionals (attached to i) on π (Moeglin-Waldspurger).

Conjecture (Hales, Moy and Prasad) $\mathcal{V}_\pi = \cup_x G_{x,d(\pi)+}$. Proved for $p >> 0$ (de Backer 2002), or p odd if $G = SL_2(F)$ (Nevins). Claim all p for $GL_2(F)$

Assume $\mathrm{char}(F) = p$ The local character expansion is valid for $G = GL_n(D)$, using the truncated exponential $x \mapsto 1 + x$ (Bertrand Lemaire).

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Nilpotent orbital integrals are not known to converge.

When p is bad there may be infinitely many nilpotent orbital integrals, or different numbers of unipotent orbits in G and nilpotent orbits in $\mathrm{Lie}(G)^*$ (even geometrically).

When $R = \mathbb{C}$ the number of nilpotent orbits is not ususally the same as the number of associated classes of parabolic subgroups so such a simple answer for the asymptotics as for $GL_n(D)$ cannot hold.

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We looked at $SL_2(F)$ to see what happens in that basic case.

The nilpotent orbits of $SL_2(F)$ are represented by the matrices $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ of orbit determined by $x(F^*)^2$. So

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> π extends to an open subgroup H of $GL_2(F)$ containing the centre Z of $GL_2(F)$.

When $\mathrm{char}(F) \neq 2$, can take $H = SL_2(F)Z$ open of finite index in G, that index being $F^*/(F^*)^2$.

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To analyse irreducible representations of $SL_2(F)$ we may replace $SL_2(F)$ by H

- which has only finitely many nilpotent orbits,
- the (complex) nilpotent orbital integrals converge,
- any irreducible smooth complex representation π of H has a local character expansion (Bertrand Lemaire).
- There are virtual smooth $\mathbb{C}$-representations $\pi_1, \dots, \pi_N$ of H such any irreducible smooth $\mathbb{C}$-representation of H is equal to

$$\pi \sim c_\pi(0)\mathbf{1} + \sum_{i=1}^{N} c_\pi(i)\pi_i, \quad c_\pi(i) \in \mathbb{Z}$$

Irreducible R-representations of $SL_2(F)$

Assume R algebraically closed, $\mathrm{char}(R) \neq p$. Write $H' = H_{\det=1}$ for any subgroup H of $GL_2(F)$.

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• Classification of the irreducible smooth R-representations of $SL_2(F)$ and a local Langlands R-correspondence for $SL_2(F)$

• Given (π, V) in an L-packet $L(\Pi)$ of length $|L(\Pi)|$, there is an integer $c_\pi(0) \in \mathbb{Z}$ and an L-packet $\pi_1, \ldots, \pi_4$ such that

$$\pi = c_\pi(0)\, \mathbf{1} + \sum_{i=1}^{4/|L(\Pi)|} \pi_i$$

on some open pro-p subgroup of $SL_2(F)$.

• We have for large integers $i >> 0$,
$|L(\Pi)| \dim V^{I'_i} = |L(\Pi)| \dim V^{I'_{i+1/2}} = -a_\Pi + 2q^i$.

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$$\begin{cases} -a_\Pi + (q+1)q^{i-1} & \text{if } \Pi|_{GL_2(F)_{val(detg)\in 2\mathbb{Z}}} \text{ irreducible} \\ -a_\Pi + 2bq^{i-1} & \text{otherwise} \end{cases}$$

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with $b = 1$ or q depending on the parity of i and on the component of $\Pi|_{ZK_0SL_2(F)} = \Pi^+ \oplus \Pi^-$ containing π.

When p is odd, there is only one L-packet $L(\Pi)$ of length 4. We have $a_\Pi = -2$. For (π, V)in this L-packet,

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So if the asymptotics of the dimensions of fixed points by congruence $G_{x,r+i}$ subgroups of a parahoric subgroup are polynomial, then the coefficients of the polynom are rational numbers, not always integers. The asymptotics may depend on the parity of i.

What can be true ?

When $\mathrm{char}(F) = 0$ or p large enough (p is good for G, or does not divide the order of the absolute Weyl group of G).
There should be a finite number of virtual smooth R-representations π_i such that any irreducible smooth R-representation π of G coincides on some open pro-p subgroup of G with an integral linear combination of the π_i's

$$\pi = \sum_i c_\pi(i)\pi_i, \quad c_\pi(i) \in \mathbb{Z} \quad \text{on } G_{d(\pi)+}$$

When $\mathrm{char}(F) = p$ is not large enough, the same should be true provided we consider representations with bounded depth.