Representations of reductive p-adic groups

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What is a reductive p-adic group ?

Why are their representations useful ?

Why are their representations interesting ?

The p-adic group $GL_2(\mathbb{Q}_p)$ is an example of a reductive p-adic group

p a prime number

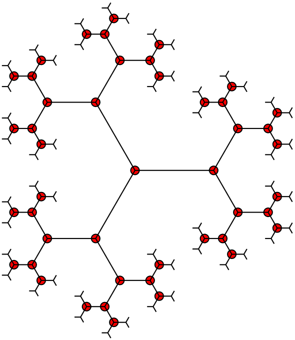
$GL_2(\mathbb{Q}_p)$ has a countable basis of open compact subgroups

$$GL_2(\mathbb{Z}_p)) \supset \mathrm{Id}+pM_2(\mathbb{Z}_p) \supset \mathrm{Id}+p^2M_2(\mathbb{Z}_p)) \supset \ldots \supset \mathrm{Id}+p^iM_2(\mathbb{Z}_p)) \supset \ldots$$

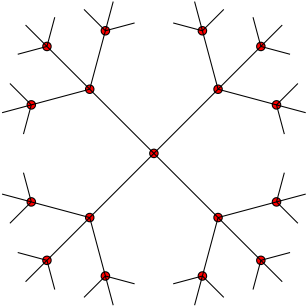
$GL_2(\mathbb{Q}_p)$ is locally a pro-p group

The $p + 1$-regular tree

$p = 2$



$p = 3$



The vertices are the homothety classes $[L]$ of lattices $L = \mathbb{Z}_p e \oplus \mathbb{Z}_p f$ of $\mathbb{Q}_p^2$. The edges $[L] - [L']$ for lattices such that $pL \subsetneq L' \subsetneq L$. The group $GL(2, \mathbb{Q}_p)$ acts naturally on the tree.

The maximal compact open subgroup $GL_2(\mathbb{Z}_p)$ fixes the vertex $[\mathbb{Z}_p \oplus \mathbb{Z}_p]$.

The Iwahori group $I = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}^*$ fixes each point in the edge $[\mathbb{Z}_p \oplus \mathbb{Z}_p] - [\mathbb{Z}_p \oplus p\mathbb{Z}_p]$.

$$\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}^* \supset \mathrm{Id} + \begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p \end{pmatrix} \supset \mathrm{Id} + p\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \supset \ldots$$

$$\supset \ldots\ldots \supset \mathrm{Id} + p^i\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \supset \mathrm{Id} + p^i\begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p \end{pmatrix} \supset \ldots$$

They are the Moy-Prasad filtrations of $GL_2(\mathbb{Z}_p)$ and of I.

Other Examples of reductive p-adic groups

There are finite extensions F of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$, of any degree (it is not like $\mathbb{R}$ or $\mathbb{C}$). Note $\mathrm{char}(F) = 0$ or p.

A reductive p-adic group is the group $\underline{G}(F)$ of F-rational points of a connected reductive F-group.
For the topology induced by F, $\underline{G}(F)$ is locally a pro-p group.

F^*, D^* for a quaternion F-algebra D, $GL_2(D)$.
$GL_n(F), SL_n(F), \ldots$ for $n \geq 1$, symplectic groups, orthogonal groups, unitary groups, groups of exceptional types

The motivation to study reductive p-adic groups is arithmetic

Langland's bridge

The local Langlands conjectures relate the absolute Galois group Gal_F of F with the reductive p-adic groups $\underline{G}(F)$.

The absolute Galois group Gal_F is the Galois group of a separable closure F^{sep}/F.
Gal_F contains a lot of information on the arithmetic of F.

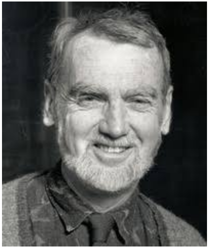
Gal_F is a compact topological group equal to the projective limit of the Galois groups of the finite Galois sub-extensions F'/F. The ramification subgroups (upper numerotation) form a countable basis of open subgroups.

Gal_F and $\underline{G}(F)$ are locally pro-p groups.

A generalisation of local class field theory

local class field theory : description of abelian extensions of F via 1-dimensional representations of the Weil group W_F, a cousin of Gal_F)

$$W_F^{ab} \simeq F^*$$



Robert P.Langlands 1936 -

Local Langlands conjectures: description of higher dimensional representations of W_F in terms of representations of matrices.

Harmonic analysis to prove conjectures in Number Theory

More on the Langlands's bridge in Tasho Kaletha's course.

To study Gal_F one studies the representations of reductive p-adic groups where many tools are available.

You will learn those tools in the Arizona winter school 2025.

Tool 1: The Bruhat-Tits building and the Moy-Prasad filtrations of the parahoric subgroups

Alan Moy

Gopal Prasad 1945-

More in Jessica Fintzen's course

It is time now to introduce representations !

Groups are invisible objects that "we see" only through their linear actions on spaces, called representations.

Let R be a field. An R-representation of a group G is an R-vector space V with a linear action of G

$$\pi : G \to \mathrm{Aut}_R(V).$$

The irreducible complex representations of the finite groups of Lie type $\underline{G}(\mathbb{F}_q)$, for instance $GL_2(\mathbb{F}_p)$, have finite dimension and can be constructed geometrically (Deligne and Lusztig).

Tool 2: The irreducible complex representations of the finite groups of Lie type

More in Charlotte Chan's course.

When G is locally a pro-p group (Gal_F or a reductive p-adic group) one supposes (π, V) smooth i.e. continuous: the fixator of any $v \in V$ is open in G

$$V = \cup_K V^K, \quad K \text{ compact open subgroup of } G$$

The irreducible smooth R-representations of Gal_F are finite dimensional because Gal_F is compact. But

The irreducible smooth representations of a reductive p-adic group are rarely finite dimensional

Example: The trivial representation is the only finite dimensional irreducible smooth representations of $SL_n(\mathbb{Q}_p)$.

Admissible replaces finite dimension

(π, V) is admissible if $\dim V^K < \infty$ for any o.c.sg K of G

If $\mathrm{char}(R) \neq p$ or $G = GL_2(\mathbb{Q}_p)$, irreducible $\Rightarrow$ admissible

Tool 3: If $\mathrm{char}(R) \neq p$, the Haar R-measure on G.

Alfred Haar 1885-1933

$$\mathrm{vol}(\mathrm{Id} + p^i M_2(\mathbb{Z}_p), dg) =$$

$$\mathrm{vol}(GL_2(\mathbb{Z}_p), dg) p^{-i+1} |GL_2(\mathbb{F}_p|^{-1}$$

A character for an infinite dimensional representation ?

The trace of a finite dimensional R-representation (π, V) of G is the function

$$\mathrm{tr}(\pi) : G \to R, \qquad g \mapsto \mathrm{tr}(\pi(g)),$$

called the character of π. It carries a lot of information.

When (π, V) is infinite dimensional but admissible

$$V = \cup_K V^K, \ \ \dim V^K < \infty$$

and there is a Haar R-measure dg on G, the linear map

$$\mathrm{tr}(\pi) : C_c^\infty(G, R) \to R, \qquad f \mapsto \mathrm{tr}(\int_G f(g)\pi(g)\, dg).$$

is called the distribution character of π

$C_c^\infty(G, R)$ is the space of compactly supported functions $f : G \to R$ bi-invariant by some K. So the endomorphism $\pi(fdg) = \int_G f(g)\pi(g)dg$ of V has image in V^K.

Harmonic analysis

If $\mathrm{char}(F) = 0$ and $R = \mathbb{C}$, the distribution character is represented by a locally integrable G-invariant function $\theta_\pi(g)$ on G, locally constant on G_{reg}

$$\mathrm{tr}(\pi(fdg)) = \int_G f(g)\theta_\pi(g)\,dg.$$

Harish-Chandra 1923-1983

The character of π is the locally constant function

$$\theta_\pi : G_{rs} \to \mathbb{C}$$

.

If π has infinite dimension, then θ_π is not locally constant near the identity. The local expansion of θ_π around the identity is another deep theorem leading to many questions with recent partial answers More in my second lecture !

Characters are an important guide towards a deeper understanding of the local Langlands correspondence.

Tool 4 : If $\mathrm{char}(F) = 0$, characters of admissible smooth $\mathbb{C}$-representations

You will learn more on characters in Tasho Kaletha's course.

Where one constructs representations of G by induction from representations of smaller subgroups

Tool 5: Smooth parabolic induction and compact induction

The example of $G = GL_2(\mathbb{Q}_p)$

• **Smooth parabolic induction** : T diagonal group, B triangular group

$$T \leftarrow B \rightarrow G$$

Start from a smooth R-representation (σ, W) of T, inflate to B, induce to get the smooth R-representation

$$\mathrm{ind}_{B,T}^{G}\sigma$$

of G by right translation on the space of functions

$$\{f : GL_2(\mathbb{Q}_p) \rightarrow W \quad f(bgk) = \tilde{\sigma}(b)(f(g)) \ \ b \in B, g \in G, k \in K_f\}$$

for some open compact subgroup K_f of G.

• **Compact induction**

$$K = \mathbb{Q}_p^* \, GL_2(\mathbb{Z}_p) \;\text{ or }\; \mathbb{Q}_p^* \, \langle I, \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \rangle$$

(representing the conjugacy classes of open compact mod center sg of G containing the center)

Start from a smooth R-representation (σ, W) of K and compactly induce to G to get the representation of G

$$\mathrm{ind}_K^G \sigma$$

by right translation on the space of functions
$\{f : GL_2(\mathbb{Q}_p) \to W, f(kg) = \sigma(k)(f(g)), \; k \in K, g \in G\}$,
supported on a finite union of classes Kh

Families of irreducible representations

One family Irreducible subquotients of $\mathrm{ind}_{B,T}^{G}\chi$ for the smooth characters $\chi : T \to R^*$.

Supercuspidal: irreducible smooth not a subquotient of a parabolically induced

This ordering of irreducible smooth R-representations of $GL_2(\mathbb{Q}_p)$ generalises to any reductive p-adic group G

A lasting conjecture proved in many cases If $\mathrm{char}(R) \neq p$, the supercuspidal R-representations are compactly induced representations $\mathrm{ind}_K^G(\sigma)$ from irreducible smooth R-representations σ of K open subgroup containing the center, compact mod center.

Roger Howe 1945-

$G = GL_2(F)$

Philip Kutzko 1946-

J.K. Yu

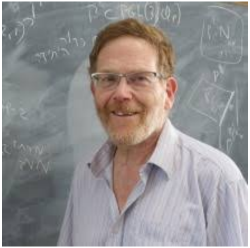
More in Jessica Fintzen and in Charlotte Chan course.

Assume $\mathrm{char}(R) = p$

Irrreducible does not imply admissible, no Haar R-measure on G. $\mathrm{ind}_K^G \sigma$ has infinite length or is 0.

Parabolic induction is simpler ! Supercuspidal R-representations of $GL_2(\mathbb{Q}_p)$ are classified.

Laure Barthel



Ron Livne

Christophe Breuil

There is an endomorphism T of cokernel

$$\mathrm{ind}_K^G \mathrm{Sym}^k(\mathbb{F}_p^2)/(T) \quad \text{irreducible supercuspidal, } 0 \leq k \leq p-1.$$

But the supercuspidal R-representations of a general G remain totally mysterious. The mathematicians try to understand them since 25 years.

New tools are needed when $\mathrm{char}(R) = p$

More on in Herzig's course !

Tool 6 : Hecke algebras

For our open compact subgroup K of G, the free abelian group $\mathbb{Z}[K\backslash G/K]$ carries a structure of an associative ring, called a Hecke ring

$$\mathbb{Z}[K\backslash G/K] \simeq \mathrm{End}_G \mathrm{ind}_K^G 1$$

For any smooth R-representation of (π, V),

$$V^K \simeq \mathrm{Hom}_K(1, V) \simeq \mathrm{Hom}_G(\mathrm{ind}_K^G 1, V)$$

is a $R[K\backslash G/K]$-module.

When K is an open compact subgroup of G of pro-ordrer invertible R (hence $\mathrm{char}(R) \neq p$), the map $V \to V^K$ is a bijection between

- the irreducible smooth R-representations (π, V) of G with $V^K \neq 0$
- the irreducible $R[K\backslash G/K]$-modules

What are the properties of the algebras $R[K\backslash G/K]$? Is it easy to classify their irreducible modules ?

Finiteness Theorem 2024

$\mathbb{Z}[1/p][K\backslash G/K]$ is a finitely generated module over its center, and the center is a finitely generated $\mathbb{Z}[1/p]$-algebra.

Jean-Francois Dat

David Helm

Rob Kurinczuk

Gil Moss

using the excursions operators of Fargues-Scholze.
It is very possible that inverting p is not necessary.

When $K = I$ is an Iwahori subgroup, $\mathbb{Z}[I\backslash G/I]$ satisfies the finiteness theorem, it is an affine Hecke ring, has been well studied, and plays an important role in the theory of smooth representations of G.

When K is a special parahoric subgroup, $\mathbb{Z}[K\backslash G/K]$ is commutative finitely generated (the Satake isomorphism over $\mathbb{Z}$).

Langlands interpreted the Satake isomorphism over $\mathbb{C}$ and in a particular case, as a parametrization of irreducible smooth $\mathbb{C}$-representations of G with a non-zero K-invariant vectors by semisimple conjugacy classes in a complex group "dual" to G. He used the parametrization to define (partial) L-functions for automorphic representations of adelic reductive groups, and with the dual group he formulated a conjectural classification of all irreducible smooth $\mathbb{C}$-representations of G (the local Langlands correspondence).