AWS 2025: Characters of representations of reductive p-adic groups

Lecturer: Tasho Kaletha
Project Assistant: Jialiang Zou

1 Overview of the material

To a finite-dimensional complex-valued representation (π, V) of a group G one can associate the character function

$$\Theta : G \to \mathbb{C}, \qquad \Theta(g) = \mathrm{tr}(\pi(g)).$$

This function carries a lot of information about the representation. A basic fact in the representation theory of *finite* (and more generally *compact topological*) groups is that every irreducible representation is finite-dimensional and its isomorphism class is determined by the character function.

Real and p-adic Lie groups, being generally neither finite nor compact, have very few interesting finite-dimensional representations. In this setting one focuses on so-called "admissible" representations, which even-though infinite-dimensional, have a robust theory of characters developed by Harish-Chandra. To such a representation (π, V) one first defines the character as a distribution

$$\Theta : \mathcal{C}_c^\infty(G) \to \mathbb{C}, \qquad \Theta(g) = \mathrm{tr}(\pi(f)),$$

where $\pi(f) : V \to V$ is the operator $\pi(f)v = \int_G f(g)\pi(g)v dg$. A fundamental result of Harish-Chandra is that this distribution is represented by a smooth function

$$\Theta : G_{\mathrm{rs}} \to \mathbb{C},$$

which, while not originally defined in the same way as in the finite-dimensional case, behaves in analogous ways.

Characters enter the representation theory of real and p-adic reductive groups in many different ways. For example, they play a key role in the classification of representations of real reductive groups; a corresponding theory in the p-adic case is still not available. Nonetheless, they are an important guide towards both explicit constructions and a deeper understanding of the local Langlands correspondence. They are also the main terms in the spectral side of the (local and global) Arthur-Selberg trace formulas, and in this way enter the study of automorphic forms.

2 Plan of lectures

2.1 The characters of discrete series representations

In this lecture we will review recent progress on the character formula for supercuspidal representations of p-adic reductive groups. The guiding lights are

1. Hairsh-Chandra's character formula for discrete series representations of real reductive groups.

2. The Deligne-Lusztig character formula for finite groups of Lie type.

3. Springer's hypothesis, proved by Kazhdan.

For 1. it is known that discrete series representations of a real reductive group G are parameterized by pairs (S,θ) consisting of an elliptic maximal torus S (unique up to conjugation by $G(\mathbb{R})$ if it exists) and a character θ of $S(\mathbb{R})$ with regular differential. Harish-Chandra's formula then states that for regular elements $s \in S(\mathbb{R})$ one has

$$\Theta_{\pi_{(S,\theta)}} = (-1)^{q(G)} \sum_{w \in W_{\mathbb{R}}(S,G)} \frac{\theta(s^w)}{\prod_{\alpha>0}(1-\alpha(s^w)^{-1})}.$$

The denominator is known as the "Weyl denominator", whose absolute value is well understood, and whose complex phase is rather subtle.

For 2., a Deligne-Lusztig virtual representation for a finite group of Lie type is given by a torus character pair (S,θ), and its character at any group element $g = su$ has the formula

$$\Theta_{\pi_{(S,\theta)}} = \sum_{\substack{x \in G/C_G(s) \\ xsx^{-1} \in S}} \theta(xsx^{-1}) Q_S^{C_G(xsx^{-1})}(u).$$

For 3., it was conjectured by Springer and showed by Kazhdan that the Green function Q_S^G can be expressed as the Fourier transform of a regular semi-simple orbital integral.

Guided by these lights we will discuss the developments in [AS08], [DR09], [DS18], [Spi18], [Kal19b], [Spi21], [FKS21], which lead to a rather explicit formula for the character of a supercuspidal representation of a p-adic group G that is associated to a torus-character pair (S,θ), which includes the cases of regular supercuspidal representations.

2.2 L-packets and stable characters

In this lecture we will discuss a variation of the notion of a character, called "stable character", which is intimately related to Langlands' ideas about harmonica analysis and representation theory. A stable character is a linear combination of irreducible characters that has a strong conjugation-invariance property.

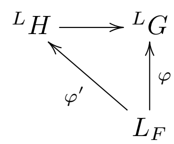
The point of departure is the basic form of the local Langlands correspondence, which should in particular partition the set of irreducible admissible representations into a disjoint union of finite sets, called L-packets. Restricting attention to tempered representations, an L-packet should have the property of "atomic stability", which means that such a set should support a stable character, unique up to scalar. Conversely, by linear independence, the stable character determines the L-packet.

This leads to the possibility to uniquely characterize the local Langlands correspondence by providing a formula for the stable character of each L-packet. We will discuss this approach, following [Kal19a]. This involves the construction of certain natural double covers of (the topological groups of rational points of) tori over local fields, as well as their L-groups (which turn out to be generally non-split extensions of the Galois group of complex tori).

2.3 Endoscopic character identities

In this lecture we will delve deeper into the discussion of L-packets, and the distinction between Harish-Chandra's notion of a character and Langlands' notion of a stable character. The bridge between these two notions is furnished by the theory of endoscopy. We will introduce the basic concepts of that theory and show how the individual characters of the representations in each tempered L-packet can be recovered from the stable characters of related L-packets.

The main relationship is captured by a diagram of the shape



where G is the group of interest, H is the endoscopic group, and L_F is the Langlands group of the local field F (a close cousin of the Galois group). This is an example of the general framework of "functoriality", a philosophy that in this case can be made quite precise, and postulates an identity between a stable character for the L-packet $\Pi_{\varphi'}$ on the group H and a certain virtual character for the L-packet Π_φ on the group G. The virtual characters arising this way provide an alternative basis for the span of the characters of the members of Π_φ and this allows one to reduce classical harmonic analysis to "stable" harmonic analysis.

The above diagram is written in "classical form" and is not quite canonical. The theory becomes cleaner if one introduces double covers of reductive groups, generalizing the double covers of tori that played a role in the character formulas. The "true" endoscopic group for G is then a certain double cover $H(F)_\pm$ of $H(F)$, and the L-packet $\Pi_{\varphi'}$ that is in endoscopic relation with Π_φ consists of genuine representations of $H(F)_\pm$.

2.4 The spectral side of the stable trace formula

In this lecture we will sketch how the discussion of characters and their stable analogs fits into the Arthur-Selberg trace formula. The setting here is that of a reductive group G over a global field F.

The trace formula is very roughly an identity of the form

$$\sum_O J_O(f) = \sum_\chi J_\chi(f),$$

where the dominant terms in the sum on the left (the so-called geometric side) are the integrals over conjugacy classes, and the dominant terms on the right (the so-called spectral side) are traces of discrete automorphic representations of G. For many applications a refined version of this formula is required, that takes the form

$$\sum_O S_O(f) = \sum_\chi S_\chi(f),$$

where the dominant terms on the left are now integrals over geometric conjugacy classes, and the dominant terms on the right are stable characters.

The passage from the first identity to the second identity is referred to as the "stabilization" of the trace formula. The stabilization of the geometric side involves the Fundamental Lemma, and more generally the transfer of orbital integrals. The stabilization of the spectral side involves the character identities discussed in the previous lecture.

We will sketch this process, focusing mainly on ansisotropic groups, where the formulas contain only the dominant terms and the ideas come forward most naturally.

3 Prerequisites and background

3.1 Prerequisites

Familiarity with local fields, basic representation theory, and reductive groups, is essential for the course. Recommended references are [Ser79], [Ser77], and [Bor91], and many suitable alternative references exist.

3.2 Background about the lectures

The following surveys will form most of the basis of the lectures: [KT22a], [KT22b], [Kal22b], [Kal]. We will use material from the following papers, among others: [Kal19b], [Kal19a], [Kal22a]. Langlands' first paper on the stable trace formula remains a timeless classic: [Lan83], and so do Kottwitz's papers [Kot84], [Kot86].

4 Projects

The projects will explore the connections between different character formulas that occur in the literature, phenomena that occur for the prime $p = 2$, and the behavior of double covers of tori in the wildly ramified setting. The techniques will come from representation theory, harmonic analysis, and the arithmetic and Galois theory of local fields.

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