CHARACTERS OF REPRESENTATIONS OF REDUCTIVE p-ADIC GROUPS

Notes for the Arizona Winter School 2025

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1 Basics on characters

To a finite-dimensional complex-valued representation (π, V) of a group G one can associate the character function

$$\Theta : G \to \mathbb{C}, \qquad \Theta(g) = \mathrm{tr}(\pi(g)).$$

This function carries a lot of information about the representation. A basic fact in the representation theory of *finite* (and more generally *compact topological*) groups is that every irreducible representation is finite-dimensional and its isomorphism class is determined by the character function.

Real and p-adic Lie groups, being generally non-compact, have very few interesting finite-dimensional representations. In this setting one focuses on so-called "admissible" representations, which even-though infinite-dimensional, have a robust theory of characters developed by Harish-Chandra.

1.1 Smooth and admissible representations

We let F denote a non-archimedean local field of characteristic zero and G the F-points of a connected reductive F-group. Let (π, V) be a complex representation of G, i.e. a (usually infinite-dimensional) complex vector space V and a group homomorphism $\pi : G \to \mathrm{GL}(V)$. We do not endow V with a topology. Instead, the continuity property that we require is the following:

Definition 1.1.1. The representation V is called *smooth*, if for each $v \in V$ the stabilizer $G_v = \{g \in G | \pi(g)v = v\}$ is open in G.

Since every open subgroup of G contains an open compact subgroup of G, we can also express this definition as

$$V = \bigcup_K V^K,$$

where the union runs over all compact open subgroups $K \subset G$ and V^K is the subspace of K-fixed vectors in G, i.e. those $v \in V$ for which $K \subset G_v$.

For character theory one needs a further property, which plays the role of finite-dimensionality.

Definition 1.1.2. The representation V is called *admissible*, if for each compact open subgroup $K \subset G$ the subspace V^K is finite-dimensional.

Fact 1.1.3. *Every irreducible smooth representation is admissible.*

1.2 Haar measures

The group G is a unimodular locally compact group, hence by Haar's theorem there exists a measure on it that is both left and right invariant, and this measure is unique up to scalar multiple. There are various ways to normalize this measure, but for now we will not have to worry about this, and will simply fix an arbitrary normalization.

We note that Haar's theorem is actually an overkill in this situation. Recall that for a real manifold the natural integrands are volume forms, rather than functions. To integrate such a form one uses a partition of unity and charts to bring down the form to an open subset of $\mathbb{R}^n$, where it can be integrated using the Lebesgue integral.

The same remains true on a F-adic analytic manifold, except that now one doesn't have a natural measure on F^n. But one can get such a measure by fixing a Haar measure on F, for example the unique one giving O_F volume 1.

In the case of G we can easily produce a volume form – pick an arbitrary alternating form of top degree on the Lie algebra and use translations (say on the left) to move it to all the points of G and thereby obtain a (left-) invariant volume form. Since the product of a function with a volume form is another volume form, fixing one volume form gives rise to an integration functional

$$\mathcal{C}_c^\infty(G) \to \mathbb{C},$$

which turns out to be a Haar integral. Moreover, it turns out that G is unimodular.

1.3 The Hecke algebra

Let $\mathcal{H} := \mathcal{H}(G) := \mathcal{C}_c^\infty(G)$ denote the complex vector space of functions $f : G \to \mathbb{C}$ that are compactly supported and locally constant. This space is generated by the characteristic functions $\mathbf{1}_{KxK}$ of K-double cosets of compact open subgroups K. In fact, for any $f \in \mathcal{H}$ we can find an open compact subgroup $K \subset G$ such that f is K-biinvariant ($f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$) and hence $f = \sum_i \mathbf{1}_{Kx_iK}$ for finitely many $x_i \in G$.

The vector space $\mathcal{H}$ becomes an algebra (the so-called "big" Hecke algebra) under convolution:

$$(f_1 * f_2)(x) = \int_G f_1(g) f_2(g^{-1}x) dg,$$

where dg is a fixed Haar measure on G. Of course the algebra structure depends on the normalization. But this is not an issue, since the algebra is not trying to have a unit. For each compact open subgroup $K \subset G$, the element $e_K = \mathrm{vol}(K, dg)^{-1} \mathbf{1}_K \in \mathcal{H}$ is an idempotent. The algebra $\mathcal{H}_K = \mathcal{C}_c(K \backslash G / K) = e_K * \mathcal{H} * e_K$ has e_K as its unit. For any sequence $K_n \subset G$ of such groups with $K_{n+1} \subset K_n$ and $\bigcap K_n = \{1\}$ the sequence $(e_{K_n})_n$ of idempotents is an approximate unit.

Fact 1.3.1. *1. The functor*

$$\{G - reps\} \to \{\mathcal{H} - mod\}$$

described above is an equivalence of categories between the category of smooth G-representations and the category of smooth $\mathcal{H}$-modules, i.e. those $\mathcal{H}$-modules V for which $\mathcal{H} * V = V$.

2. *The functor $V \mapsto V^K$ induces a bijection between the irreducible G-representations with $V^K \neq \{0\}$ and the irreducible $\mathcal{H}_K$-modules.*

1.4 Distributions

Let V be a finite-dimensional F-vector space and let $\mathcal{C}_c^\infty(V)$ be the vector space of functions $f : V \to \mathbb{C}$ that are compactly supported and locally constant.

Definition 1.4.1. A *distribution* is a linear form $d : \mathcal{C}_c^\infty(V) \to \mathbb{C}$.

Note that we are not endowing $\mathcal{C}_c^\infty(V)$ with a topology, and are not requiring any continuity property of d.

We will write $\mathcal{D}(V)$ for the space of distributions.

Let $L^1_{\text{loc}}(V)$ denote the space of functions $\phi : V \to \mathbb{C}$ that lie in $L^1(C)$ for any compact subset $C \subset V$. Such a function gives a distribution by

$$d_\phi(f) = \int_V \phi(v) f(v) dv,$$

where dv is a chosen Haar measure. In this way we obtain an injective homomorphism of vector spaces

$$L^1_{\text{loc}}(V) \to \mathcal{D}(V).$$

Definition 1.4.2. We say that the distribution $d \in D(V)$ is *representable* by a function, if it lies in the image of the above injective homomorphism.

1.5 The distribution character

Given a smooth representation (π, V) of G we can obtain a representation of the algebra $\mathcal{H}$ on V by

$$\pi(f)v = \int_G f(g)\pi(g) v dg.$$

Note that, if K is a compact open subgroup for which f is left-invariant, then $f = e_K * f$ and hence $\pi(f) \in V^K$. Thus, when V is admissible, the operator $\pi(f)$ has finite rank, and therefore also has a trace.

Definition 1.5.1. The *distribution character* of π is the distribution

$$\Theta_\pi : \mathcal{H} \to \mathbb{C}, \qquad \Theta(f) = \text{tr}(\pi(f)).$$

Here the word distribution simply means a linear functional on the complex vector space $\mathcal{H}$, without assuming any continuity property.

Remark 1.5.2. The distribution Θ_π depends on the choice of Haar measure – it scales proportionally with it.

Fact 1.5.3. *1. Let $\pi_1, \dots, \pi_n$ be pairwise non-isomorphic irreducible admissible representations. Then the distributions $\Theta_{\pi_1}, \dots, \Theta_{\pi_n}$ are linearly independent.*

2. *Two irreducible admissible representations are isomorphic if and only if their distribution characters are equal.*

3. *The distribution character is additive along exact sequences: If $0 \to \pi_1 \to \cdots \to \pi_n \to 0$ is an exact sequence of admissible representations, then $\sum_i (-1)^i \Theta_{\pi_i} = 0$.*

1.6 Harish-Chandra's representability theorem

Let $G_{\mathrm{rs}} \subset G$ denote the set of regular semi-simple elements, i.e. elements which centralize (hence are contained in) a unique maximal torus. Given such $x \in G$ contained in the maximal torus T, the *Weyl discriminant* is defined as $D_G(x) = \prod_\alpha (1 - \alpha(x))$, the product being taken over all roots of T in G. Alternatively, one can define $D_G(x)$ as the coefficient of degree rank(G) in the polynomial $\det(t - \mathrm{Ad}(x) + 1)$, and the regular semi-simple elements are precisely those with $D_G(x) \neq 0$.

Theorem 1.6.1 (Harish-Chandra)**.** *Let (π, V) be an irreducible admissible representation. The distribution Θ_π is represented by a (necessarily unique) function $\Theta_\pi : G_{rs} \to \mathbb{C}$ that is*

1. *locally constant on G_{rs},*
2. *locally integrable on G,*
3. *bounded when multiplied by the Weyl discriminant $|D_G(x)|_F^{1/2}$,*

in the sense that for all $f \in \mathcal{H}$

$$\Theta_\pi(f) = \int_G f(g) \Theta_\pi(g) dg.$$

The proof of this theorem is given in [HC99] or [HC70].

1.7 The Fourier transform

Let $\Lambda : F \to \mathbb{C}^\times$ be a non-trivial character. If V and V^* are finite-dimensional F-vector spaces in duality we can define the Fourier transform

$$\mathcal{C}_c^\infty(V) \to \mathcal{C}_c^\infty(V^*), \qquad f \mapsto \widehat{f} = \widehat{f}_{\Lambda,dx}, \qquad \widehat{f}_{\Lambda,dx}(\xi) = \int_V f(x) \Lambda(\langle x, \xi \rangle) dx,$$

depending on the choices of Λ and a Haar measure dx.

Switching the roles of V and V^* we also have the Fourier transform that assigns to $f^* \in \mathcal{C}_c^\infty(V^*)$ the function $\widehat{f}^*_{\Lambda,d\xi} \in \mathcal{C}_c^\infty(V)$.

Given dx there exists a unique $d\xi$, called the *dual measure* so that the Fourier-inversion formula

$$\widehat{\widehat{f}}(x) = f(-x)$$

holds.

Slogan: The smaller the support of f is, the larger the lattice $\Lambda \subset V^*$ is under which $\widehat{f}$ is invariant.

Given a distribution $d : \mathcal{C}_c^\infty(V^*) \to \mathbb{C}$ its Fourier transform $\widehat{d} : \mathcal{C}_c^\infty(V) \to \mathbb{C}$ is defined as

$$\widehat{d}(f) = d(\widehat{f}).$$

Slogan: The Fourier transforms of many naturally-occurring distributions are representable by functions.

Example 1.7.1. The Dirac delta distribution $\delta : \mathcal{C}_c^\infty(V^*) \to \mathbb{C}$ is not representable by a function, because its support is "too small". But its Fourier transform is given by

$$\widehat{\delta}(f) = \delta(\widehat{f}) = \widehat{f}(0) = \int_V f(x)\Lambda(\langle x, 0\rangle)dx = \int_V f(x) \cdot 1 dx,$$

and we see that $\widehat{\delta}$ is represented by the constant function 1 on V.

1.8 Orbital integrals

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G and let $\mathfrak{g}^*$ be the F-dual vector space of $\mathfrak{g}$.

For an element $x \in \mathfrak{g}$ or $x \in \mathfrak{g}^*$ we want to define the *orbital integral*

$$O_x(f) = \int_{\mathrm{Ad}(G)x} f(y)dy \quad \text{or} \quad O_x(f^*) = \int_{\mathrm{Ad}^*(G)x} f^*(y)dy$$

as a G-invariant distribution. It is not a-priori clear if this makes sense:

1. Does there exist a G-invariant measure on the G-orbit of x?
2. Is the restriction of the test function to this orbit ingegrable?

It turns out that the answer to both questions is positive. In the case of semi-simple elements this can be seen very easily. Recall that the element x is called *semi-simple* if its G-orbit is closed, and *nilpotent* if the closure of its G-orbit contains 0.

When x is semi-simple the stabilizer G_x is reductive, hence unimodular, so G/G_x does carry a G-invariant measure. Moreover, the intersection of the orbit of x with the support of f is compact, so f is integrable on that orbit.

For the general case we first consider the dual Lie algebra and take $x \in \mathfrak{g}^*$. We will recall Kirillov's observation that $\mathrm{Ad}^*(G)x$ carries a natural symplectic structure. Indeed, the alternating form $\eta_x(z, w) = x([z, w])$ on $\mathfrak{g}$ has radical equal to $\mathfrak{g}_x = \{y \in \mathfrak{g} | \mathrm{ad}^*(y)x = 0\}$, hence descends to a symplectic form on $\mathfrak{g}/\mathfrak{g}_x$. Varying x over its orbit we obtain a G-invariant 2-form η on the manifold $\mathrm{Ad}(G)^*x$, which turns it into a symplectic G-manifold. In particular, its dimension is even, say $2d$, and then $\eta^{\wedge d}$ is a G-invariant volume form on $\mathrm{Ad}(G)^*x$.

This gives a positive answer to question 1. in the case of the dual Lie algebra $\mathfrak{g}^*$. Using a G-invariant symmetric bilinear form on $\mathfrak{g}$ (F has characteristic zero) one can transport this to $\mathfrak{g}$.

To obtain a positive answer to question 2. takes more work. One can reduce from the general setting to the nilpotent setting, where one obtains an explicit formula for the integral, see [RR96].

Now that the distributions O_x have been defined, one can consider their Fourier transforms $\widehat{O}_x$. The following result of Harish-Chandra is an important input in the proof of Theorem 1.6.1, and will also be important for our study of the function Θ_π.

Theorem 1.8.1 (Harish-Chandra). *Let $\xi \in \mathfrak{g}^*$ be regular semi-simple. The distribution $(\widehat{O}_\xi)_{\Lambda,dx}$ is represented by a function $\widehat{\mu}_{\xi,\Lambda} : \mathfrak{g} \to \mathbb{C}$ that is*

1. *locally constant on $\mathfrak{g}_{rs}$,*
2. *locally integrable on $\mathfrak{g}$,*
3. *bounded when multiplied by the Weyl discriminant $|D_\mathfrak{g}(-)|_F^{1/2}$.*

Here the Weyl discriminant $D_\mathfrak{g}$ on the Lie algebra is defined in a way similar to that for the group. One can define it as the coefficient of the polynomial $\det(t - \mathrm{ad}(x))$ in degree rank(G), which is non-zero precisely when x is regular semi-simple, i.e. contained in a unique Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, in which case it equals $\prod_\alpha \alpha(x)$, the product taken over all roots of $\mathfrak{t}$ in $\mathfrak{g}$.

The function $\widehat{\mu}_{\xi,\Lambda}$ does not depend on the choice of Haar measure dx on $\mathfrak{g}$, because this measure is used twice – once to form the Fourier transform, and once to obtain the embedding $L^1_{\text{loc}}(\mathfrak{g}) \to \mathcal{D}(\mathfrak{g})$, which cancel out.

The function $\widehat{\mu}_{\xi,\Lambda}$ does depend on the choice of Λ. Any other choice is of the form $c \cdot \Lambda(x) = \Lambda(cx)$ for some $c \in F^\times$ and one checks $\widehat{\mu}_{\xi,c\Lambda} = \widehat{\mu}_{c\xi,\Lambda}$.

Later we will find useful to renormalize the function $\widehat{\mu}$ a using the usual Weyl discriminants [DS18, Definition 2.2.8] and obtain

$$\widehat{\iota}_{X^*,\Lambda}(Y) = |D(X^*)|^{\frac{1}{2}} |D(Y)|^{\frac{1}{2}} \widehat{\mu}_{X^*,\Lambda}(Y). \tag{1.1}$$

We have $\widehat{\iota}_{\xi,c\Lambda} = \widehat{\iota}_{c\xi,\Lambda}$ for $c \in O_F^\times$.

1.9 The character of a parabolic induction

Let $P = MN \subset G$ be a parabolic subgroup and let σ be an admissible representation of M. It is known that $\pi := I_P^G(\sigma)$ is also admissible. The distribution characters are related by

$$\Theta_\pi(f) = \Theta_\sigma(f^{(P)}),$$

where $f^{(P)} \in \mathcal{H}(M)$ is the constant term of $f \in \mathcal{H}(G)$, defined by

$$f^{(P)}(m) = \delta_P(m)^{1/2} \int_U \int_K f(k^{-1}muk) dk du dm,$$

and $K \subset G$ is a compact open subgroup in good relative position to P in the sense that the Iwasawa decomposition $G = PK$ holds, the measures are synchronized so that

$$\int_G f(g)dg = \int_M \int_U \int_K f(muk)dkdudm,$$

and δ_P is the modulus character of the group P, which is not unimodular; it is given by

$$\delta_P(m) = \det(\mathrm{Ad}(m)|\mathrm{Lie}(U)).$$

It follows from this formula that the function characters are related by

$$\Theta_\pi(x) = \sum_{\substack{g\in M\backslash G \\ gxg^{-1}\in M}} \frac{F_\sigma(gxg^{-1})}{|D_{G/M}(gxg^{-1})|^{1/2}},$$

where $D_{G/M}(m) = \det(1 - \mathrm{Ad}(m)|\mathrm{Lie}(G)/\mathrm{Lie}(M))$.

1.10 The Steinberg character

The group G has a special representation, called the Steinberg representation. It is obtained as follows. Let $P_0 \subset G$ be a minimal parabolic subgroup. Consider the induced representation

$$i_{P_0}^G(\delta_{P_0}^{-1/2}) = \mathrm{Ind}_{P_0}^G \mathbf{1}_{P_0} = \mathcal{C}^\infty(P_0\backslash G),$$

where $\mathcal{C}^\infty(P_0\backslash G)$ can be identified with the subspace of $\mathcal{C}^\infty(G)$ consisting of those functions that are left-invariant under P_0, and G is acting by right translation. Every standard parabolic subgroup $P_0 \subset P$ gives the submodule $\mathcal{C}^\infty(P\backslash G)$ of $\mathcal{C}^\infty(P_0\backslash G)$. The Steinberg representation is the quotient of $\mathcal{C}^\infty(P_0\backslash G)$ by the sum (not direct) of all $\mathcal{C}^\infty(P\backslash G)$ for $P_0 \subsetneq P$, cf. [Cas73]. Call this sum Σ_0 for future reference.

It is known that the Steinberg representation is a discrete series representation that is not supercuspidal. To compute its character we can use the Borel–Serre resolution.

First, recall some basic facts about reductive groups. Each standard parabolic subgroup $P_0 \subset P$ has a unique Levi factor that contains a fixed Levi factor M_0 of P_0. If A_0 is the split center of M_0, then $R(A_0, G)$ is the relative root system of G and is equipped with a set of simple relative roots $\Delta_0(G)$ whose associated set of positive roots is determined by P_0. Since $P_0 \cap M$ is also a minimal parabolic subgroup of M_0 we have the corresponding $\Delta_0(M) \subset R(A_0, M)$ and $P \leftrightarrow \Delta_0(M)$ is a bijection between the set of standard parabolic subgroups of G and the set of subsets of $\Delta_0(G)$. We shall write $\mathrm{rk}(P) = |\Delta_0(M)$.

The Borel–Serre resolution is the following complex, which one can show to be exact

$$0 \to I_r \to I_{r-1} \to \cdots \to I_0 \to \mathrm{St} \to 0,$$

where

$$I_t = \bigoplus_{P:\mathrm{rk}(P)=t} \mathcal{C}^\infty(P\backslash G)$$

and the differential $I_{t+1} \to I_t$ is given by the sum of the maps $\mathcal{C}^\infty(P\backslash G) \to \mathcal{C}^\infty(Q\backslash G)$ for parabolic subgroups $Q \subset P$ of neighboring ranks, given by the usual restriction-of-functions map multiplied by a certain sign $\epsilon(P,Q)$ that we will ignore. Note here that $I_r = \mathcal{C}^\infty(G\backslash G)$ is the trivial representation, while $I_0 = \mathcal{C}^\infty(P_0\backslash G)$, and the cokernel $I_1 \to I_0$ is the quotient of I_0 by the sum of the images of $\mathcal{C}^\infty(P_1\backslash G)$ for all standard parabolic subgroups $P_1 \subset G$ of rank 1. Since for every standard parabolic subgroup $P \subset G$ there is some P_1 contained in P, we have $\mathcal{C}^\infty(P\backslash G) \subset \mathcal{C}^\infty(P_1\backslash G)$, and hence Σ_0 equals the sum of all $\mathcal{C}^\infty(P_1\backslash G)$, confirming the exactness at the last step.

Using the fact that the trace is additive in exact sequences, together with the formula for the character of a prabolically induced representation, we conclude that for $g \in G_{\text{rs}}$ we have

$$F_{\text{St}}(g) = (-1)^{\dim(A_0)} \sum (-1)^{\dim(A_M)} \delta_P(g)^{-\frac{1}{2}} |D_{G/M}(g)|^{-\frac{1}{2}},$$

where the sum runs over the set of pairs (M,P), with $P \subset G$ a parabolic subgroup (not necessarily standard), $M \subset P$ a Levi factor, such that $g \in M$.

1.11 Characters for $\mathbf{SL}_2$

Consider the group $G = \text{SL}_2(F)$ with F non-archimedean of odd residual characteristic. We will consider the values of characters at the split torus $T \cong F^\times$ and the various non-split tori $S \cong E^1$ for quadratic extensions E/F.

The irreducible admissible representations are as follows:

1. Parabolic induction $I_B^G(\chi)$, where $B = TU$ is the standard Borel subgroup, χ is a character of $T \cong F^\times$ that is not equal to $\delta_B^{\pm 1/2}$ and is not of order 2. It is tempered precisely when χ is unitary.

   $$\Theta(t) = \frac{\chi(t) + \chi(t^{-1})}{|t - t^{-1}|}, t \in T_{\text{rs}} = F^\times \smallsetminus \{\pm 1\}, \quad \Theta(s) = 0, s \in E^1.$$

2. The two inequivalent pieces π^+, π^- of $i_B^G(\chi)$ where $\chi \neq 1$ but $\chi^2 = 1$. They are tempered, but not discrete series. To compute $\Theta_{\pi^\pm}$ it is enough to compute the two virtual characters $\Theta_{\pi^+} \pm \Theta_{\pi^-}$. This is the most basic example of inversion of endoscopic character identities.

   The formula for $\Theta_{\pi^+} + \Theta_{\pi^-}$ is the same as above, but the formula for $\Theta_{\pi^+} - \Theta_{\pi^-}$ is more interesting: It is supported on the unique E^1 such that $\ker(\chi) = N_{E/F}(E^\times) \subset F^\times$. This is the subject of one of the projects.

   The representations π^+ and π^- are examples of what Arthur calls *elliptic* representations that are not discrete series. An elliptic representation is one whose character does not vanish on the set of elliptic regular semisimple elements. Discrete series representations certainly have this property, but non-discrete tempered representations can also have it, and π^+ and π^- are examples of such. Note that irreducible unitary parabolic inductions are not elliptic, so the magic happens when such an induction breaks into pieces.

3. The Steinberg representation (quotient module of $i_B^G(\delta_B^{-1/2}) = \mathcal{C}^\infty(B\backslash G)$, submodule of $i_B^G(\delta_B^{1/2})$). It is discrete.

   $$\Theta(t) = \frac{|t| + |t|^{-1}}{|t - t^{-1}|} - 1, t \in T_{\text{rs}} = F^\times \smallsetminus \{\pm 1\}, \quad \Theta(s) = -1, s \in E^1.$$

4. The trivial representation (submodule of $i_B^G(\delta_B^{-1/2}) = \mathcal{C}^\infty(B\backslash G)$, quotient module of $i_B^G(\delta_B^{1/2})$). It is non-tempered.

$$\Theta(g) = 1.$$

5. The regular supercuspidal representations. These are parameterized by the G-conjugacy classes of pairs (S, θ), where $S \subset G$ is a compact maximal torus, thus $S \cong E^1$ for a quadratic extension E/F, and θ is a character of S such that $\theta^2 \neq 1$. The character formula for these representations was determined by Sally–Shalika, but without proof; the proofs were somewhat recently supplied by Adler–DeBacker–Sally–Spice [ADSJS10]. When E/F is unramified the result takes the form

$$\Theta_\pi(\gamma) = \begin{cases} \frac{1}{2}\operatorname{sgn}_\epsilon\big(\operatorname{Im}_\epsilon(\gamma)\big)\dfrac{\psi(\gamma)+\psi(\gamma^{-1})}{|D_G(\gamma)|^{1/2}}\big[(-1)^{r+1} + H(\Lambda', k_\epsilon)\big] & \gamma \in T^\epsilon \setminus Z(G)T^\epsilon_{r+} \\ c_0(\pi) + H(\Lambda', k_\epsilon)\dfrac{\operatorname{sgn}_\epsilon\big(\eta^{-1}\operatorname{Im}_\epsilon(\gamma)\big)}{|D_G(\gamma)|^{1/2}} & \gamma \in T^{\epsilon,\eta}_{r+} \\ c_0(\pi) + \dfrac{1}{|D_G(\gamma)|^{1/2}} & \gamma \in A_{r+} \\ c_0(\pi) & \text{otherwise, if } \gamma \in G_{r+} \\ 0 & \text{otherwise, if } \gamma \notin G_{r+}. \end{cases}$$

and when E/F is ramified it takes the form

$$\Theta_\pi(\gamma) = \begin{cases} \dfrac{\operatorname{sgn}_\varpi\big(\operatorname{Im}_\varpi(\gamma)\big)H(\Lambda', k_\varpi)}{|D_G(\gamma)|^{1/2}}\Big\{\psi(\gamma) \ + \ \psi(\gamma^{-1})\big[\dfrac{\operatorname{sgn}_\varpi(-1)+1}{2}\big]\Big\} & \\ & \gamma \in T^\theta \setminus Z(G)T^\theta_r \\ \dfrac{q^{-1/2}}{2\,|D_G(\gamma)|^{1/2}} \displaystyle\sum_{\substack{\gamma' \in (C_\varpi)_{r:r+} \\ \gamma' \neq \gamma^{\pm 1}}} \operatorname{sgn}_\varpi\big(\operatorname{tr}_\varpi(\gamma - \gamma')\big)\psi(\gamma') & \\ \qquad + \ \dfrac{1}{2}H(\Lambda', k_\varpi)\operatorname{sgn}_\varpi\big(\eta^{-1}\operatorname{Im}_\varpi(\gamma)\big)\dfrac{\psi(\gamma)+\psi(\gamma^{-1})}{|D_G(\gamma)|^{1/2}} & \\ & \gamma \in T^{\varpi,\eta}_r \setminus T^{\varpi,\eta}_{r+} \\ \dfrac{q^{-1/2}}{2\,|D_G(\gamma)|^{1/2}} \displaystyle\sum_{\gamma' \in (C_\varpi)_{r:r+}} \operatorname{sgn}_\varpi\big(\operatorname{tr}_{\epsilon\varpi}(\gamma) - \operatorname{tr}_\varpi(\gamma')\big)\psi(\gamma') & \gamma \in T^{\epsilon\varpi,\eta}_r \setminus T^{\epsilon\varpi,\eta}_{r+} \\ c_0(\pi) + H(\Lambda', k_\varpi)\dfrac{\operatorname{sgn}_\varpi\big(\eta^{-1}\operatorname{Im}_\varpi(\gamma)\big)}{|D_G(\gamma)|^{1/2}} & \gamma \in T^{\varpi,\eta}_{r+} \\ c_0(\pi) + \dfrac{1}{|D_G(\gamma)|^{1/2}} & \gamma \in A_{r+} \\ c_0(\pi) & \text{otherwise, if } \gamma \in G_{r+} \\ 0 & \text{otherwise, if } \gamma \notin G_{r+}. \end{cases}$$

6. The non-regular supercuspidal representations. These are of depth zero, and are parameterized by the G-conjugacy classes of pairs (S, θ), where $S \subset G$ is an unramified maximal torus, and $\theta : N_G(S) \to \mathbb{C}^\times$ is a character that extends the unique unramified sign character $S = E^1 \cong k_E^1 \to \{\pm 1\}$. The formulas of Sally–Shalika, proved by Adler–DeBacker–Sally–Spice, are

$$\Theta_{\pi^\pm}(\gamma) = \begin{cases} \dfrac{\mathrm{sgn}_\varpi(\gamma+\gamma^{-1}+2)}{2}\Big\{H(\Lambda',k_\epsilon)\dfrac{\mathrm{sgn}_\epsilon\big(\mathrm{Im}_\epsilon(\gamma)\big)}{|D_G(\gamma)|^{1/2}}-1\Big\} & \gamma\in T^\epsilon\setminus Z(G)T^\epsilon_{0+} \\ \dfrac{1}{2}\Big\{\dfrac{1}{|D_G(\gamma)|^{1/2}}-1\Big\} & \gamma\in A_{0+} \\ \dfrac{1}{2}\Big\{\pm H(\Lambda',k_{\theta'})\dfrac{\mathrm{sgn}_{\theta'}\big(\eta^{-1}\,\mathrm{Im}_{\theta'}(\gamma)\big)}{|D_G(\gamma)|^{1/2}}-1\Big\} & \gamma\in T^{\theta',\eta}_{0+}, \\ 0 & \text{otherwise.} \end{cases}$$

It will be helpful to consider also the case $F=\mathbb{R}$ and compare. Thus let $G=\mathrm{SL}_2(\mathbb{R})$. The irreducible admissible representations are as follows.

1. Parabolic induction $I_B^G(\chi)$, where $B=TU$ is the standard Borel subgroup, $\chi=(\mathrm{sgn}^m,|-|^s)$ with $m\in\mathbb{Z}/2\mathbb{Z}$ and $s\in\mathbb{C}$ is a character of $T\cong\mathbb{R}^\times\cong\{\pm1\}\times\mathbb{R}_{>0}$ such that $s\notin m+1$. It is tempered precisely when χ is unitary, i.e. $s\in i\mathbb{R}$. The character formula is exactly as in the case of non-archimedean base field F.
2. The two inequivalent pieces of $i_B^G(\chi)$ where $\chi=(1,0)$. They are tempered, but not discrete series. We will denote them by $\mathcal{D}_1^+$ and $\mathcal{D}_1^-$. They are sometimes called "limits of discrete series". Their character formulas will be given below, together with the case of discrete series, because they are the same.
3. The holomorphic and antiholomorphic discrete series $\mathcal{D}_k^+,\mathcal{D}_k^-$ of weight k, for $k\in\mathbb{Z}_{\geq2}$, (quotient module of $i_B^G((k,k-1))$, submodule of $i_B^G((k,1-k))$). They are discrete. They are equivalently parameterized by the G-conjugacy classes of triples (S,C,θ), where $S\subset G$ is a compact maximal torus, thus $S\cong\mathbb{S}^1$, C is an orientation of the 1-dimensional real vector space Lie(S), and θ is a character of S whose differential is positive for the orientation. Note that, unless $d\theta=0$, i.e. θ is constant, the orientation is uniquely determined by $d\theta$. The character formulas are given by (with $k\geq1$, i.e. including limits of discrete series)
$$\Theta_k^+(k_\varphi)=\frac{-e^{i(k-1)\varphi}}{e^{i\varphi}-e^{-i\varphi}},\quad \Theta_k^-(k_\varphi)=\frac{e^{-i(k-1)\varphi}}{e^{i\varphi}-e^{-i\varphi}},\quad k_\varphi=\begin{bmatrix}\cos(\varphi) & ,\sin(\varphi)\\ -\sin(\varphi) & \cos(\varphi)\end{bmatrix},$$
$$\Theta_k^\pm(a_t)=\frac{e^{(k-1)t}(1-\mathrm{sgn}(t))+e^{-(k-1)t}(1+\mathrm{sgn}(t))}{2|e^t-e^{-t}|},\quad a_t=\begin{bmatrix}e^t & \\ & e^{-t}\end{bmatrix}.$$
4. The $(k-1)$-dimensional representation $\mathrm{Sym}^{k-2}(\mathrm{Std})$ for $k\in\mathbb{Z}_{\geq2}$ (submodule of $i_B^G((k,k-1))$, quotient module of $i_B^G((k,1-k))$). It is non-tempered.

2 The characters of regular supercuspidal representations

In this section we will focus our attention to the characters of regular supercuspidal representations. In the process we will review the definition of these representations.

2.1 The guiding light of real groups

We will review Harish-Chandra's theory of the discrete series representations of real reductive groups and their characters. This will serve as a motivation for the p-adic case.

Let G be a connected reductive $\mathbb{R}$-group.

Definition 2.1.1. An irreducible admissible representation π of G belongs to the *discrete series*, if its matrix coefficients are square-integrable functions on G.

Clearly this cannot happen unless the center of G is compact, which was the assumption Harish-Chandra made in his work, because any matrix coefficient transforms under the center by the central character of the representation. To allow for non-compact center, we have the slight variation.

Definition 2.1.2. An irreducible admissible representation π of G belongs to the *relative discrete series*, if it has a unitary central character and its matrix coefficients are square-integrable functions on G/Z. It belongs to the *essential discrete series*, if some twist of it by a character of G belongs to the relative discrete series.

Theorem 2.1.3 (Harish-Chandra)**.**

1. *G has relative discrete series representations if and only if it has an elliptic (compact modulo center) maximal torus S. Such a torus is unique up to G-conjugation.*
2. *The set of isomorphism classes of essentially discrete series representations is in bijection with the set of G-conjugacy classes of tuples (S, B, θ), where $S \subset G$ is an elliptic maximal torus, B is a $\mathbb{C}$-Borel subgroup containing S, and $\theta : S \to \mathbb{C}^\times$ is a character whose differential is B-dominant.*
3. *The representations $\pi_{(S,B,\theta)}$ is uniquely determined by the property that its Harish-Chandra character function, restricted to S, is given by the formula*

$$(-1)^{q(G)} \sum_{w \in N(S,G)/S} \frac{\theta(\gamma^w)}{\prod_{\alpha \overset{B}{>} 0} (1 - \alpha(\gamma^w)^{-1})}.$$

Remark 2.1.4.

1. For most θ the differential $d\theta$ is regular, hence determines B uniquely. Only those few θ whose differential lies on a root wall need to be supplemented by a choice of chamber.
2. There is a cleaner way to state the theorem which will be important for us later. From the pair (B, θ) we can consider $d\theta + \rho_B \in \mathrm{Lie}^*(S)$. This element is always regular, and in fact it is the infinitesimal character of $\pi_{(S,B,\theta)}$. But it need not be the differential of the character of S. However, it is the differential of a character of a canonical double cover $S_\pm$ of S. Using this double cover, we can state the classification as being by pairs $(S, \theta_\pm)$, where $\theta_\pm$ is a genuine character of $S_\pm$ with regular differential. The datum B is now superfluous. Moreover, we can write the character formula as

$$(-1)^{q(G)} \sum_{w \in N(S,G)/S} \frac{\theta_\pm(\dot\gamma^w)}{\prod\limits_{\langle d\theta_\pm, \alpha \rangle > 0} (\alpha^{1/2}(\dot\gamma^w) - \alpha^{-1/2}(\dot\gamma^w))}.$$

   Both numerator and denominator are genuine functions of $S_\pm$, so their quotient descends to S. Moreover, the denominator has a very pleasant transformation property under the Weyl group:

$$\prod_{\langle d\theta_\pm, \alpha \rangle > 0} (\alpha^{1/2}(\dot\gamma^w) - \alpha^{-1/2}(\dot\gamma^w)) = \mathrm{sgn}(w) \prod_{\langle d\theta_\pm, \alpha \rangle > 0} (\alpha^{1/2}(\dot\gamma) - \alpha^{-1/2}(\dot\gamma)).$$

   This is used actively in the theory of compact and non-compact Lie groups, for example in the proof of the Weyl character formula, and Harish-Chandra's extension of it to the non-compact setting. We will revisit this point in the p-adic setting as well.

2.2 Depth-zero regular supercuspidal representations

Let G be a connected reductive group over a non-archimedean local field F of characteristic zero. We will define and classify regular depth-zero supercuspidal representations by $G(F)$-conjugacy classes of pairs (S, θ) consisting of an elliptic maximally unramified maximal torus $S \subset G$ and a regular depth-zero character $\theta : S(F) \to \mathbb{C}^\times$. We will then compute their Harish-Chandra character functions.

2.2.1 Maximally unramified elliptic maximal tori

Fact 2.2.1. *Let $S \subset G$ be a maximal torus and $S' \subset S$ be the maximal unramified subtorus. The following statements are equivalent.*

1. *S' is of maximal dimension among the unramified subtori of G.*
2. *S' is not properly contained in an unramified subtorus of G.*
3. *S is the centralizer of S' in G.*
4. *$S \times F^u$ is a minimal Levi subgroup of $G \times F^u$.*
5. *The action of I_F on $R(S, G)$ preserves a set of positive roots.*

Proof. This follows from the fact that $G \times F^u$ is quasi-split, which is a theorem of Steinberg. □

Definition 2.2.2. A maximal torus $S \subset G$ will be called *maximally unramified* if it satisfies the above equivalent conditions.

When G splits over F^u then S is unramified. Therefore, this notion generalizes the notion of an unramified maximal torus to the case of ramified groups.

The assignments $S \mapsto S'$ and $S' \mapsto \text{Cent}(S', G)$ are mutually inverse bijections between the set of maximally unramified maximal tori of G and the set of maximal unramified tori of G. The $G(F)$-conjugacy classes of the latter were classified by DeBacker in [DeB06]. We shall now extract the main results relevant to us.

The first key point is that we can associate to S a point $x \in \mathcal{B}^{\text{red}}(G, F)$ as follows: Since $S' \subset G$ becomes a maximal split torus over F^u, we have the apartment $\mathcal{A}^{\text{red}}(S, F^u) \subset \mathcal{B}^{\text{red}}(G, F^u)$. This apartment is Frobenius-invariant, since S is defined over F, and contains Frobenius-fixed points, namely the center of mass of any (automatically finite) Frobenius-orbit. Since S is elliptic, there is in fact a unique such fixed point x.

Lemma 2.2.3. *The point x is a vertex of $\mathcal{B}^{red}(G, F)$.*

2.2.2 Parahoric integral models

The point provides the chain of subgroups

$$G(F)_x^0 \subset G(F)_x^1 \subset G(F)_x \subset G(F).$$

Here $G(F)_x^*$ is the stabilizer of the point x for the action of $G(F)^*$. We recall that $G(F)^1$ denotes the intersection of the kernels of the group homomorphisms $\mathrm{ord} \circ \chi : G(F) \to \mathbb{Z}$ for all F-rational characters $\chi : G \to \mathbb{G}_m$, while $G(F)^0$ denotes the kernel of the Kottwitz homomorphism $G(F) \to \pi_1(G)_I^{\mathrm{Fr}}$.

Each of these subgroups turns out to be the group of integral points of a smooth integral model of G, namely $\mathscr{G}_x^*(O_F) = G(F)_{x,*}$, where

$$\mathscr{G}_x^0 \to \mathscr{G}_x^1 \to \mathscr{G}_x.$$

The two maps are open and closed immersions and realize $\mathscr{G}_x^0$ as the relative identity component of either of the other models. The models $\mathscr{G}_x^0$ and $\mathscr{G}_x^1$ are affine, but $\mathscr{G}_x$ need not be.

We shall write G_x^* for the reductive quotient of the special fiber of the integral model $\mathscr{G}_x^*$. Then G_x^0 coincides with the identity components of G_x^1 and G_x. In particular, G_x^1 is a usually disconnected affine algebraic group over k_F with reductive neutral connected component, and G_x is a smooth k_F-group scheme with reductive identity component. We have $\pi_0(\mathsf{G}_x) = \pi_1(G)_I^{\mathrm{Fr}}$ and $\pi_0(\mathsf{G}_x^1) = (\pi_1(G)_I^{\mathrm{Fr}})_{\mathrm{tor}}$.

The reduction map $G(F)_x^* = \mathscr{G}_x^*(O_F) \to \mathscr{G}_x^*(k_F)$ is surjective (smoothness) and furthermore the projection map $\mathscr{G}_x^*(k_F) \to \mathsf{G}_x^*(k_F)$ is also surjective (the kernel is a connected unipotent group, whose Galois cohomology vanishes).

Lemma 2.2.4. *1. The special fiber of the (automatically connected) ft-Neron model of S' embeds canonically as an elliptic maximal torus S' of the reductive group G_x°. Explicitly, $\mathsf{S}'(k_{F'}) \subset \mathsf{G}_x^\circ(k_{F'})$ is the image in $G(F')_{x,0:0+}$ of $S(F') \cap G(F')_{x,0}$, or equivalently of $S'(F') \cap G(F')_{x,0}$, for every unramified extension F'.*

2. *Every elliptic maximal torus of G_x° arises in this way.*

3. *Let $S_1, S_2 \subset G$ be two maximally unramified elliptic maximal tori. Assume that their points in $\mathcal{B}^{red}(G,F)$ coincide, call them x. Assume furthermore that $S_1(F^u) \cap G(F^u)_{x,0}$ and $S_2(F^u) \cap G(F^u)_{x,0}$ have the same projection to $\mathsf{G}_x^\circ(\overline{k_F})$. Then S_1 and S_2 are $G(F)_{x,0+}$-conjugate.*

Lemma 2.2.5. *Let $S \subset G$ be a maximally unramified elliptic maximal torus with associated point $x \in \mathcal{B}^{red}(G,F)$. Then*

$$S(F) \cap G(F)_{x,0} = S(F)_0.$$

The group G_x is not affine and is highly disconnected, but one still has good control over it. Namely, the center Z_G of the p-adic group projects to give a closed subgroup $\mathsf{Z} \subset \mathsf{G}_x$ with the property that $[\mathsf{Z} \cdot \mathsf{G}_x^\circ : \mathsf{G}_x] < \infty$. Moreover, $\pi_0(\mathsf{G}_x)$ is a finitely generated abelian group.

The p-adic torus S also projects to give a closed subgroup $\mathsf{S} \subset \mathsf{G}_x$, and we have $\mathsf{S} = \mathsf{Z} \cdot \mathsf{S}^\circ$, where S° can be understood as the following equal subgroups: the identity component of S, the intersection $\mathsf{S} \cap \mathsf{G}_x^\circ$, and ft-Neron model of the maximal unramified subtorus $S' \subset S$.

2.2.3 Depth zero characters

Let G° be a connected reductive group defined over a finite field k, let $\mathsf{S}' \subset \mathsf{G}^\circ$ be a maximal torus, and let $\bar\theta : \mathsf{S}'(k) \to \bar{\mathbb{Q}}_l^\times$ be a character. In [DL76, Definition 5.15], Deligne and Lusztig define two regularity conditions for a character $\bar\theta : \mathsf{S}'(k) \to \bar{\mathbb{Q}}_l^\times$, which we shall now recall. They say that $\bar\theta$ is *in general position*, if its stabilizer in $\Omega(\mathsf{S}', \mathsf{G}^\circ)(k_F)$ is trivial. They say that $\bar\theta$ is *non-singular*, if it is *not orthogonal to any coroot*. We will not review what this means, but rather state an equivalent criterion.

Lemma 2.2.6. *Let k' be a finite extension of k splitting S', $\bar\theta : \mathsf{S}'(k) \to \bar{\mathbb{Q}}_l^\times$ a character, and $\alpha^\vee \in R^\vee(\mathsf{S}', \mathsf{G}^\circ)$. Then $\bar\theta$ is orthogonal to $\alpha^\vee$ if and only if the character $\bar\theta \circ N \circ \alpha^\vee : k'^\times \to \bar{\mathbb{Q}}_l^\times$ is trivial. In particular, $\bar\theta$ is non-singular if and only if for each $\alpha^\vee \in R^\vee(\mathsf{S}', \mathsf{G}^\circ)$ the character $\bar\theta \circ N \circ \alpha^\vee : k'^\times \to \bar{\mathbb{Q}}_l^\times$ is non-trivial.*

We will now define a third regularity condition on $\bar\theta$. We say that $\bar\theta$ is absolutely regular, if for some (hence any) finite extension k' of k splitting S' the character $\bar\theta \circ N$ has trivial stabilizer in $\Omega(\mathsf{S}', \mathsf{G}^\circ)$. It is clear that absolutely regular implies general position. By [DL76, Corollary 5.18] general position implies non-singular.

Lemma 2.2.7. *If the center of G° is connected, then the notions of non-singular, general position, and absolutely regular, are equivalent.*

As discussed earlier, Bruhat–Tits theory leads to groups G that are not necessarily connected or even affine, but they satisfy the following finiteness properties:

Assumption 2.2.8.

1. *$\pi_0(\mathsf{G})$ is a finitely generated abelian group,*
2. *there is a closed central subgroup $\mathsf{Z} \subset \mathsf{G}$ such that $[\mathsf{Z} \cdot \mathsf{G}^\circ : \mathsf{G}] < \infty$.*

In this setting we let $\mathsf{S} = \mathsf{Z} \cdot \mathsf{S}'$ and we write $N(\mathsf{S}, \mathsf{G})$ as usual for the normalizer of S in G and $\Omega(\mathsf{S}, \mathsf{G}) = N(\mathsf{S}, \mathsf{G})/\mathsf{S}$.

Definition 2.2.9. We shall call $\bar\theta : \mathsf{S}(k) \to \bar{\mathbb{Q}}_l^\times$ (or $\bar\theta : \mathsf{S}(k) \to \mathbb{C}^\times$) *regular* (resp. *extra regular*) if the stabilizer of $\bar\theta|_{\mathsf{S}^\circ(k)}$ in $N(\mathsf{S}, \mathsf{G})(k)$ (resp. $\Omega(\mathsf{S}, \mathsf{G})(k)$) is trivial.

Fact 2.2.10. *We have*

$$\bar\theta \text{ extra regular } \Rightarrow \bar\theta \text{ regular } \Rightarrow \bar\theta|_{\mathsf{S}^\circ(k)} \text{ in general position} .$$

If the point of $\mathcal{B}^{red}(G, F)$ associated to S is superspecial, then the converse implications also hold.

2.2.4 Deligne-Lusztig induction in the disconnected setting

Let (Z, G) satisfy Assumptions 2.2.8. Let $\mathsf{S}' \subset \mathsf{G}^\circ$ be an elliptic maximal torus, set $\mathsf{S} = \mathsf{Z} \cdot \mathsf{S}'$, and let $\bar\theta : \mathsf{S}(k) \to \bar{\mathbb{Q}}_l^\times$ be a regular character.

A choice of a Borel $\bar k$-subgroup $\mathsf{B} = \mathsf{S}'\mathsf{U}$ containing S' leads to the Deligne–Lusztig variety

$$Y = \{g \in \mathsf{G} | g^{-1}F(g) \in \mathsf{U} \cdot F(\mathsf{U})\},$$

defined just as in the case of connected groups.

Lemma 2.2.11. *The cohomology $H_c^*(Y, \bar{\mathbb{Q}}_l)_{\bar{\theta}}$ vanishes away from the middle degree, and in that degree provides an irreducible cuspidal representation $\kappa_{(\mathsf{S},\bar{\theta})}$ of $\mathsf{G}(k)$.*

The meaning of cuspidal here is that the restriction to $\mathsf{G}^\circ(k)$, which is a finite length representation, has cuspidal irreducible constituents.

2.2.5 Classification of regular depth-zero supercuspidal representations

We now come to the definition and construction of regular depth-zero supercuspidal representations. Let π be an irreducible supercuspidal representation of $G(F)$ of depth zero. According to [MP96, Proposition 6.8] there exists a vertex $x \in \mathcal{B}^{\text{red}}(G, F)$ such that the restriction $\pi|_{G(F)_{x,0}}$ contains the inflation to $G(F)_{x,0}$ of an irreducible cuspidal representation κ of $G(F)_{x,0:0+}$.

Definition 2.2.12. We shall call π *regular* (resp. *extra regular*) if κ is a Deligne-Lusztig cuspidal representation associated to an elliptic maximal torus S' of G_x° and a character $\bar{\theta} : \mathsf{S}'(k_F) \to \mathbb{C}^\times$ that is regular (resp. extra regular) in the sense of Definition 2.2.9.

Note that, since $\mathsf{S}'(k_F)$ is a finite group, $\bar{\theta}$ takes values in $\bar{\mathbb{Q}}^\times$, so replacing $\bar{\mathbb{Q}}_l$ with $\mathbb{C}$ is inconsequential.

Proposition 2.2.13.
1. *Let $S \subset G$ be a maximally unramified maximal torus and let $\theta : S(F) \to \mathbb{C}^\times$ be a depth-zero character whose factorization $\bar{\theta} : \mathsf{S}(k) \to \mathbb{C}^\times$ is regular. Let $\kappa_{(S,\theta)}$ be the irreducible cuspidal representation of Lemma 2.2.11. Then $\pi_{(S,\theta)} = c\text{-}Ind_{S(F)G(F)_{x,0}}^{G(F)} \kappa_{(S,\theta)}$ is irreducible, and hence supercuspidal, regular, of depth zero.*

2. *Every regular depth-zero supercuspidal representation of $G(F)$ is of the form $\pi_{(S,\theta)}$ for some maximally unramified elliptic maximal torus S and regular depth-zero character $\theta : S(F) \to \mathbb{C}^\times$.*

3. *Two representations $\pi_{(S_1,\theta_1)}$ and $\pi_{(S_2,\theta_2)}$ are isomorphic if and only if the pairs (S_1, θ_1) and (S_2, θ_2) are $G(F)$-conjugate.*

2.2.6 The character of regular depth-zero supercuspidal representations

The first step is to compute the character of the representation $\kappa_{(\mathsf{S},\bar{\theta})}$. This can be done using the arguments in Deligne–Lusztig, adapted to the disconnected setting. We first need a version of the topological Jordan decomposition.

Lemma 2.2.14. *Assume that G splits over a tame extension. Let $\gamma \in G(F)_{x,0} \cdot S(F)$ be a semi-simple element. Then $\gamma = \gamma_s\gamma_u$, where $\gamma_s \in G(F)_{x,0} \cdot S(F)$ is topologically semi-simple modulo A_G and $\gamma_u \in G(F)_{x,0}$ is topologically unipotent. Both γ_s and γ_u are unique up to multiplication by elements of $A_G(F)_{0+}$. The image of the decomposition $\gamma = \gamma_s\gamma_u$ in $\bar{\mathsf{G}}_x(k_F)$ is the usual Jordan decomposition in the (possibly disconnected) finite group of Lie type $\bar{\mathsf{G}}_x$. If T is a maximal torus containing γ, then $\gamma_s, \gamma_u \in T(F)$. In particular, γ_s and γ_u commute.*

Proposition 2.2.15. *Assume that G splits over a tamely ramified extension. The character of $\kappa_{(S,\theta)}$ at a semi-simple element $\gamma \in \mathsf{G}(k)$ is given by the formula*

$$(-1)^{r_G - r_S} |C(\gamma_s)^\circ(k_F)|^{-1} \sum_{\substack{h \in \mathsf{G}_x^\circ(k_F) \\ h^{-1}\gamma_s h \in \mathsf{S}(k_F)}} \theta(h^{-1}\gamma_s h) Q_{h\mathsf{S}'h^{-1}}^{C(\gamma_s)^\circ}(\gamma_u),$$

where $C(\gamma_s) \subset \mathsf{G}_x$ is the centralizer of γ_s, and r_G and r_S are the split ranks of G and S, respectively.

The function Q is a Green function. Note that it is used only for a connected reductive group, even though G itself is disconnected. That is because only the semi-simple part γ_s sees the disconnectedness, while γ_u lies in G_x°. This in turn has to do with the assumption that G splits over a tame extension, which guarantees that $\pi_1(G)_I$, and hence $\pi_0(\mathsf{G}_x)$, does not have an element of order p.

The Green function is in general difficult to compute. But Springer had conjectured that it can be expressed as the Fourier transform of an orbital integral. This conjecture was proved by Kazhdan. Combining Kazhdan's result with the above formula and the arguments in [DR09] one obtains the following formula for the character of $\pi_{(S,\theta)}$.

Theorem 2.2.16. *Assume that G splits over a tamely ramified extension. The character of $\pi_{(S,\theta)}$ at a semi-simple element $\gamma \in G(F)$ is given by the formula*

$$(-1)^{rk_F(G)-rk_F(J)} \sum_{\substack{g \in S(F)\backslash G(F)/J(F) \\ {}^g\gamma_s \in S(F)}} \theta({}^g\gamma_s)\widehat{\iota}^J_{X^g}(\log \gamma_u),$$

where $J = C(\gamma_s)^\circ$ and X is any element of Lie$^*(S)(F)_0$ whose projection to Lie$^*(S)(F)_{0:0+}$ is regular.

Corollary 2.2.17. *If $\gamma \in G(F)$ is regular semi-simple and its image in $G_{ad}(F)$ is topologically semi-simple, then the character of $\pi_{(S,\theta)}$ at γ is zero unless γ is ($G(F)$-conjugate to) an element of $S(F)$, in which case it is given by the formula*

$$(-1)^{r_G - r_S} \sum_{w \in N(S,G)(F)/S(F)} \theta(\gamma^w),$$

where again r_G and r_S are the split ranks of G and S, respectively.

2.3 Positive-depth toral representations

Consider now a connected reductive F-group G. We will define and classify positive-depth toral supercuspidal representations by $G(F)$-conjugacy classes of pairs (S,θ) consisting of an elliptic tamely ramified maximal torus $S \subset G$ and a generic positive-depth character $\theta : S(F) \to \mathbb{C}^\times$.

2.3.1 Generic characters

The definition of a generic character of a given positive depth goes back to Adler and Yu. In Yu's language there are two conditions, called GE1 and GE2, that must be satisfied. Yu shows that GE2 is implied by GE1 when p is not a torsion prime for the dual root datum of G (in particular, if p does not divide the order of the Weyl group).

We will give here an equivalent formulation, that is very analogous to Lemma 2.2.6.

Lemma 2.3.1. *Let $\theta : S(F) \to \mathbb{C}^\times$ be a character that is trivial on $S(F)_{r+}$ for some $r > 0$. Let E/F be the splitting extension of S.*

1. *The character θ satisfies GE1 for depth r if and only if, for each $\alpha \in R(S, G)$,*
$$\theta \circ N_{E/F} \circ \alpha(E_r^\times) \neq 1.$$
2. *Given GE1, the character θ satisfies GE2 for depth r if and only if the stabilizer of $\theta|_{S(F)_r}$ in $\Omega(S, G)(F)$ is trivial.*

2.3.2 Adler's construction

Consider a pair (S, θ), with S elliptic tamely ramified and θ generic of depth $r > 0$. The theorem of tame descent in Bruhat–Tits theory due to Rousseau and Prasad can be used to produce from S a point $x \in \mathcal{B}^{\text{red}}(G)$, just like in the case when S is maximally unramified. However, x need not be a vertex any more.

We have the Moy–Prasad filtration groups $G(F)_{x,a}$ for $a \geq 0$, as well as the Lie algebra lattices $\mathfrak{g}(F)_{x,a}$ for any $a \in \mathbb{R}$, and the Moy-Prasad isomorphism
$$G(F)_{x,a}/G(F)_{x,b} \to \mathfrak{g}(F)_{x,a}/\mathfrak{g}(F)_{x,b}, \qquad 0 < a \leq b \leq 2a.$$
We have the decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$ that is defined over F, stable under the adjoint action of S, and compatible with formation of Moy-Prasad lattices.

Set $s = r/2$. Via the isomorphism $S(F)_{s+}/S(F)_{r+} \to \mathfrak{s}(F)_{s+}/\mathfrak{s}(F)_{r+}$ we transport θ to a character of $\mathfrak{s}(F)_{s+}/\mathfrak{s}(F)_{r+}$ and use the direct sum decomposition
$$\mathfrak{g}(F)_{x,s+}/\mathfrak{g}(F)_{x,r+} = \mathfrak{s}(F)_{x,s+}/\mathfrak{s}(F)_{x,r+} \oplus \mathfrak{n}(F)_{x,s+}/\mathfrak{n}(F)_{x,r+}$$
to extend this character trivially on the second summand. We can then use the Moy–Prasad isomorphism for G to obtain a character of $\widehat{\theta}$ of $G(F)_{x,s+}/G(F)_{x,r+}$. It turns out that the intertwining group of this character is $S(F)G(F)_{x,s}$. This means that, in order to obtain an irreducible representation, we need to first obtain from $\widehat{\theta}$ an irreducible representation κ of $S(F)G(F)_{x,s}$, and then compactly induce to $G(F)$. This compact induction will be automatically irreducible, hence supercuspidal.

If $G(F)_{x,s} = G(F)_{x,s+}$ we obtain κ simply by combining the character θ on $S(F)$ with the character $\widehat{\theta}$ of $G(F)_{x,s}$. But if $G(F)_{x,s} \neq G(F)_{x,s+}$, the situation is more complicated. One uses the theory of the Weil-Heisenberg representation to obtain κ.

We will now describe how this works. Consider
$$J_+ = (S, G)_{x,r,s+} \subset J = (S, G)_{x,r,s} \subset K = SG_{x,s}.$$
One has the exact sequence
$$1 \to J_+ \to J \to J/J_+ \to 1,$$
and via the Moy-Prasad isomorphism J/J_+ is seen to be an $\mathbb{F}_p$-vector space with a natural symplectic structure. Pushing out this extension under (the restriction of) $\widehat{\theta}$ one obtain an extension of the symplectic vector space by $\mathbb{F}_p$,

which is understood as a Heisenberg group. The theory of the Heisenberg representation provides a unique representation $\widehat{\theta}^H$ of J with $\widehat{\theta}$-isotypic restriction to J_+.

So far so good, but we actually need a representation of $K = S(F)G(F)_{x,s}$. We can consider $J \rtimes S(F)$ for the conjugation action of $S(F)$ on J and have the following exact sequence

$$1 \to S(F)_r \to J \rtimes S(F) \to K \to 1,$$

so our goal is to produce from $\widehat{\theta}^H$ a representation of $J \rtimes S(F)$ that kills the anti-diagonal embedding of $S(F)_r$ and thus descends to K.

The action of $S(F)$ on J preserves the isomorphism class of $\widehat{\theta}^H$, so we immediately obtain an extension of $\tilde{\theta}$ to a *projective* representation of $J \rtimes S(F)$. But we don't know if this representation linearizes, and even if we did know that, that won't give us a particular linearization (in general different linearizations differ by character twists).

Just as a quick aside: The action of $S(F)_{0+}$ on J/J_+ is trivial, so we can extend $\tilde{\theta}$ trivially to $J \rtimes S(F)_{0+}$. But the extension to $J \rtimes S(F)$ is subtle.

To obtain a particilar realization, one notes that the action of $S(F)$ on the quotient J/J_+ preserves the natural symplectic structure on this vector space, which gives a homomorphism of $S(F)$ into the symplectic group of that space. The theory of the Weil–Heisenberg representation now extends $\widehat{\theta}^H$ to a representation $\widehat{\theta}^W$ of $J \rtimes S(F)$ on the same underlying vector space. The result, being an extension of $\widehat{\theta}^H$, still has $\widehat{\theta}$-isotypic restriction to $J \rtimes \{1\}$, while it has trivial restriction to $\{1\} \rtimes S(F)_{0+}$. We then pull θ under the projection on the second factor and form $J \rtimes S(F)$ that restricts trivially to the anti-diagonal embedding of $S(F)_r$, hence descends to K. This is the construction of κ when $G(F)_{x,s+} \neq G(F)_{x,s}$.

2.3.3 The Adler–Spice character formula

In their paper [AS09], Adler–Spice compute the Harish-Chandra character of the toral representations $\pi_{(S,\theta)}$. In fact, their theorem can handle slightly more representations, but we will not dwell on that. The computation is somewhat involved: one has to compute the character of the Weil–Heisenberg representation and the Harish-Chandra integral character formula, each of which produces roots of unity which have to be carefully collected. These roots of unity were later reinterpreted in the paper [DS18] of DeBacker–Spice. The resulting formula is this.

Theorem 2.3.2 (Adler-Spice)**.** *Let $\gamma \in G(F)$ be regular semi-simple and let $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$ be a normal r-approximation. Then $\Theta_{\pi_{(S,\theta)}}(\gamma)$ equals*

$$\sum_{\substack{g \in J(F)\backslash G(F)/S(F) \\ \gamma^g_{<r} \in S(F)}} \epsilon_{\mathit{sym,ram}}(\gamma^g_{<r})\epsilon^{\mathit{ram}}(\gamma^g_{<r})\tilde{e}(\gamma^g_{<r}) \cdot \theta(\gamma^g_{<r})\widehat{\iota}^J_{g_X}(\log(\gamma_{\geq r})) \qquad (2.1)$$

Note that, structurally, the result is very similar to that of Theorem 2.2.16, but the roots of unity that appear are quite different. In order to make sense of this formula we need to explain the roots of unity that appear in it.

We need to explain the notation. Fix again a non-trivial character $\Lambda : F \to \mathbb{C}^\times$, with the additional assumption that Λ is trivial on $\mathfrak{p}_F$ but non-trivial on O_F. Let $X \in \mathrm{Lie}^*(S)(F)_{-r}$ be a G-generic element that realizes the character θ. We abbreviate $\gamma^g_{<r} = g^{-1}\gamma_{<r}g$ and ${}^gX = \mathrm{Ad}^*(g)X$. Setting $J = \mathrm{Cent}(\gamma_{<r}, G)^\circ$, the condition $\gamma^g_{<r}$ on the summation index g implies that $\mathrm{Ad}(g)S$ is a subgroup of J and in particular ${}^gX \in \mathfrak{j}^*(F)$. Therefore, the function $\widehat{\iota}_{\mathfrak{j},{}^gX}$ that represents the normalized Fourier-transform of the integral along the coadjoint orbit of ${}^gX$ in $\mathfrak{j}^*(F)$ makes sense. Moreover, since both the function itself and the element X now depend on the choice of Λ in a parallel way, the entire expression $\widehat{\iota}_{\mathfrak{j},{}^gX}$ is independent of Λ. The map log is either the true logarithm function, provided it converges at $\gamma_{\geq r}$, or else the inverse of a mock-exponential map [AS09, Appendix A].

The remaining objects in the formula: $\epsilon_{\mathrm{sym,ram}}$, ϵ^{ram}, and $\tilde{e}$, are all complex roots of unity of order dividing 4 and will be the focus of our study. We shall now give their definition following [DS18, §4.3]. Let T be a maximal torus of G^{d-1} containing $\gamma^g_{<r}$ and such that $x \in \mathcal{A}^{\mathrm{red}}(T, E)^\Gamma$ for some finite Galois extension E/F splitting T. We consider the following subset of the real numbers, defined for each $\alpha \in R(T, G)$ by

$$\mathrm{ord}_x(\alpha) = \{r \in \mathbb{R} | \mathfrak{g}_\alpha(F_\alpha)_{x,r+} \neq \mathfrak{g}_\alpha(F_\alpha)_{x,r}\},$$

where we have abbreviated by $\mathfrak{g}_\alpha(F_\alpha)_{x,r}$ the intersection $\mathfrak{g}_\alpha(F_\alpha) \cap \mathfrak{g}(F_\alpha)_{x,r}$. Based on this set we define the following subsets of the root system $R(T, G)$

$$\begin{aligned} R_{\gamma^g_{<r}} &= \{\alpha \in R(T,G) \smallsetminus R(T, G^{d-1}) | \alpha(\gamma^g_{<r}) \neq 1\}, \\ R_{r/2} &= \{\alpha \in R_{\gamma^g_{<r}} | r \in 2\mathrm{ord}_x(\alpha)\}, \\ R_{(r-\mathrm{ord}_{\gamma^g_{<r}})/2} &= \{\alpha \in R_{\gamma^g_{<r}} | r - \mathrm{ord}(\alpha(\gamma^g_{<r}) - 1) \in 2\mathrm{ord}_x(\alpha)\}. \end{aligned}$$

For $\alpha \in R_{(r-\mathrm{ord}_{\gamma^g_{<r}})/2}$ symmetric and ramified we define

$$t_\alpha = \frac{1}{2} e_\alpha N_{F_\alpha/F_{\pm\alpha}}(w_\alpha) \langle d\alpha^\vee(1), X\rangle (\alpha(\gamma^g_{<r}) - 1) \in O^\times_{F_\alpha}.$$

Here e_α is the ramification degree of F_α/F and $w_\alpha \in F^\times_\alpha$ is any element of valuation $(\mathrm{ord}(\alpha(\gamma_{<r}) - 1) - r)/2$. The existence of w_α is argued in the proof of [AS09, Proposition 5.2.13]. Finally, we introduce the Gauss sum

$$\mathfrak{G} = q^{-1/2} \sum_{x \in k_F} \Lambda(x^2) \in \mathbb{C}^\times.$$

With this notation at hand, we come to the definition of the three roots of unity.

$$\epsilon_{\mathrm{sym,ram}}(\gamma^g_{<r}) = \prod_{\alpha \in \Gamma \backslash (R_{(r-\mathrm{ord}_{\gamma^g_{<r}})/2})_{\mathrm{symm,ram}}} \mathrm{sgn}_{F_{\pm\alpha}}(G_{\pm\alpha})(-\mathfrak{G})^{f_\alpha} \mathrm{sgn}_{k^\times_{F_\alpha}}(t_\alpha). \quad (2.2)$$

The product here runs over the Γ-orbits of symmetric ramified roots belonging to $R_{(r-\mathrm{ord}(\gamma^g_{<r}))/2}$. For each such α, let $G_{\pm\alpha}$ be the subgroup of G generated by the root subgroups for the two roots α and $-\alpha$. It is a semi-simple group of rank 1 defined over $F_{\pm\alpha}$, and $\mathrm{sgn}_{F_{\pm\alpha}}$ denotes its Kottwitz sign [Kot83], which equals 1 if the $G_{\pm\alpha}$ is split and -1 if it is anisotropic. Furthermore, f_α is the

degree of the field extension k_{F_α}/k_F, and $\mathrm{sgn}_{k_{F_\alpha}^\times}$ is the quadratic character of the cyclic group $k_{F_\alpha}^\times$, onto which we can project the element $t_\alpha \in O_{F_\alpha}^\times$. Both $\mathfrak{G}$ and t_α depend on the choice of Λ (the latter through X) and it is easy to check that this dependence cancels out.

$$\epsilon^{\mathrm{ram}}(\gamma^g_{<r}) = \prod_{\alpha \in \Gamma\times\{\pm 1\}\backslash (R_{r/2})^{\mathrm{sym}}} \mathrm{sgn}_{k_{F_\alpha}^\times}(\alpha(\gamma^g_{<r})) \cdot \prod_{\alpha\in\Gamma\backslash(R_{r/2})_{\mathrm{sym,unram}}} \mathrm{sgn}_{k^1_{F_\alpha}}(\alpha(\gamma^g_{<r})). \quad (2.3)$$

Here the superscript sym means that we are taking $\Gamma \times \{\pm 1\}$-orbits of asymmetric roots, while the subscripts sym,unram mean that we are taking Γ-orbits of roots that are symmetric and unramified. In the first product, we project $\alpha(\gamma^g_{<r}) \in O_{F_\alpha}^\times$ to $k_{F_\alpha}^\times$. In the second product, the $F_\alpha/F_{\pm\alpha}$-norm of the element $\alpha(\gamma^g_{<r}) \in O_{F_\alpha}^\times$ is trivial, because the root α is symmetric. The same is true for the projection of $\alpha(\gamma^g_{<r})$ to $k_{F_\alpha}^\times$, because the symmetric root α is unramified. The group $k^1_{F_\alpha}$ of elements of $k_{F_\alpha}^\times$ with trivial $k_{F_\alpha}/k_{F_{\pm\alpha}}$-norm is cyclic and we apply its quadratic character to the projection of $\alpha(\gamma^g_{<r})$. Finally

$$\tilde{e}(\gamma^g_{<r}) = \prod_{\alpha\in\Gamma\backslash(R_{(r-\mathrm{ord}_{\gamma^g_{<r}})/2})_{\mathrm{sym}}} (-1). \quad (2.4)$$

Note that while the original definition of $\tilde{e}$ does not contain the subscript sym, we may restrict the product to symmetric roots by [DS18, Remark 4.3.4].

Each of these signs implicitly depends on T, but their product is independent of T.

2.4 Reinterpreting the roots of unity

The structural similarity of the formulas in Theorem 2.2.16 and 2.3.2 begs the question of whether they can be brought into a unified form. This is indeed possible, but requires a careful study of the roots of unity that occur in Theorem 2.3.2. The key result is the following.

Proposition 2.4.1. *Let $T \subset G$ be a maximal torus with a tamely ramified splitting field E/F and let $x \in \mathcal{B}^{red}(G,F)\cap\mathcal{A}^{red}(T,E)$. For any $\alpha \in R(T,G)_{sym}$ we have*

$$\mathrm{ord}_x(\alpha) = \begin{cases} e_\alpha^{-1}\mathbb{Z}, & \text{if } \alpha \text{ is ramified} \\ e_\alpha^{-1}\mathbb{Z}, & \text{if } \alpha \text{ is unramified and } f_{(G,T)}(\alpha) = +1 \\ e_\alpha^{-1}(\mathbb{Z}+\frac{1}{2}), & \text{if } \alpha \text{ is unramified and } f_{(G,T)}(\alpha) = -1 \end{cases}$$

where $f_{(G,T)}(\alpha)$ is the toral invariant defined in [Kal15, §4.1].

Once this is known, one can make the following definition:

Definition 2.4.2. We say that $\alpha \in R(S,G)$ is *symmetric* if $-\alpha \in \Gamma\cdot\alpha$. For such α define

1. $\Gamma_\alpha = \mathrm{Stab}(\alpha,\Gamma)$, $\Gamma_{\pm\alpha} = \mathrm{Stab}(\{\pm\alpha\},\Gamma)$, $F_\alpha = E^{\Gamma_\alpha}$, $F_{\pm\alpha} = E^{\Gamma_{\pm\alpha}}$,
2. $a_\alpha = \langle H_\alpha, X\rangle \in F_\alpha$

3. $\chi'_\alpha : F_\alpha^\times \to \mathbb{C}^\times$ the unique character which is specified as follows:

   (a) If $F_\alpha/F_{\pm\alpha}$ is unramified, then let χ'_α be the unique unramified quadratic character.

   (b) If $F_\alpha/F_{\pm\alpha}$ is ramified, then let χ'_α be the unique character that lifts the quadratic character of the residue field $k_\alpha^\times$, and moreover satisfies $\chi'_\alpha(2a_\alpha) = \lambda_{F_\alpha/F_{\pm\alpha}}(\Lambda \circ \mathrm{tr}_{F_{\pm\alpha}/F})$.

With this, we can define the function

$$\Delta_{II}^{\mathrm{abs}}[a,\chi'] : S(F) \to \mathbb{C}, \qquad s \mapsto \prod_{\substack{\alpha \in R(S,G)_{\mathrm{sym}}/\Gamma \\ \alpha(s) \neq 1}} \chi'_\alpha\left(\frac{\alpha(\gamma)-1}{a_\alpha}\right).$$

Proposition 2.4.3. *The value of the normalized character of the toral supercuspidal representation $\pi_{(S,\theta)}$ at the element $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$ is given as the product*

$$e(G)e(J)\epsilon_L(X^*(T_G)_\mathbb{C} - X^*(T_J)_\mathbb{C}, \Lambda) \cdot \sum_{\substack{g \in J(F)\backslash G(F)/S(F) \\ \gamma^g_{<r} \in S(F)}} \Delta_{II}^{abs}[a,\chi'](\gamma^g_{<r})\epsilon_{f,ram}(\gamma^g_{<r})\epsilon^{ram}(\gamma^g_{<r})\theta(\gamma^g_{<r})\widehat{\iota}_{\mathfrak{j},{}^gX^*}(\log(\gamma_{\geq r}))$$

On the other hand, in the depth-zero case, we can take $a_\alpha \in F_\alpha$ to be an arbitrary unit, subject to $a_{-\alpha} = -a_\alpha$, and χ_α to be the quadratic unramified character.

Proposition 2.4.4. *The value of the normalized character of the regular depth-zero supercuspidal representation $\pi_{(S,\theta)}$ at the element $\gamma = \gamma_s \cdot \gamma_u$ is given as the product*

$$e(G)e(J)\epsilon_L(X^*(T_G)_\mathbb{C} - X^*(T_J)_\mathbb{C}, \Lambda) \cdot \sum_{\substack{g \in J(F)\backslash G(F)/S(F) \\ \gamma^g_s \in S(F)}} \Delta_{II}^{abs}[a,\chi'](\gamma^g_s)\theta(\gamma^g_s)\widehat{\iota}_{\mathfrak{j},{}^gX^*}(\log(\gamma_u))$$

The two formulas now look quite closely aligned, were it not for the occurrence of the term $\epsilon_{f,\mathrm{ram}}(\gamma^g_{<r})\epsilon^{\mathrm{ram}}(\gamma^g_{<r})$ in the toral case. While studying $\epsilon^{\mathrm{ram}}(\gamma^g_{<r})$, Spice discovered that Yu's paper had an error, which broke a number of key proofs. The error was traced to a misprint in the paper [Gér77] on the Weil representation. It was shown by Fintzen that, nonetheless, Yu's construction does still produce supercuspidal representations. But the failure of the technical results in Yu's paper impeded the computation of characters beyond the toral case. In [FKS21] a modification of Yu's construction was proposed. This modified construction leads to a different map $(S,\theta) \mapsto \pi_{(S,\theta)}$, for which the following variant of the above result holds.

Proposition 2.4.5. *The value of the normalized character of the toral supercuspidal representation $\pi_{(S,\theta)}$ at the element $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$ is given as the product*

$$e(G)e(J)\epsilon_L(X^*(T_G)_\mathbb{C} - X^*(T_J)_\mathbb{C}, \Lambda) \cdot \sum_{\substack{g \in J(F)\backslash G(F)/S(F) \\ \gamma^g_{<r} \in S(F)}} \Delta_{II}^{abs}[a,\chi'](\gamma^g_{<r})\theta(\gamma^g_{<r})\widehat{\iota}_{\mathfrak{j},{}^gX^*}(\log(\gamma_{\geq r}))$$

Thus, finally, the formulas in the regular depth-zero and toral cases obtain the same structure.

2.5 Regular supercuspidal representations

Consider now a connected reductive F-group G. We will define and classify regular supercuspidal representations, which generalize the regular depth-zero supercuspidal representations and the toral supercuspidal representations. They are classified by $G(F)$-conjugacy classes of pairs (S, θ) consisting of an elliptic tamely ramified maximal torus $S \subset G$ and a regular character $\theta : S(F) \to \mathbb{C}^\times$ of arbitrary depth (but not assumed generic). We will then discuss their Harish-Chandra characters.

2.5.1 Yu's construction and regular supercuspidal representations

We recall that Yu's construction is a map

$$\left\{ \begin{array}{c} (G^0 \subset G^1 \subset \cdots \subset G^d = G) \\ \pi_{-1} \\ (\phi_0, \phi_1, \dots, \phi_d) \end{array} \right\} \xrightarrow{\text{J.K.Yu}} \{\text{irred. s.c reps of } G(F)\}$$

It is a theorem of Julee Kim, strengthened by Jessica Fintzen, that this construction is surjective, i.e. produces all supercuspidal representations when p is not too small ($p \nmid |W|$ in Fintzen's stronger version).

The datum consists of a tower of tame elliptic twisted Levi subgroups, a depth-zero supercuspidal representation π_{-1} of G^0, and a sequence of characters, where $\phi_i : G^i \to \mathbb{C}^\times$ is G^{i+1}-generic for $i < d$.

The datum can be thus taken as a label for the representation, but different labels can lead to the same representation, i.e. the map is not injective. Its fibers wre described in [HM08] by an explicit equivalence relation, called "refactorization".

Definition 2.5.1. The representation π is *(extra)regular*, if π_{-1} is such in the sense of Definition 2.2.12.

We note that the regularity of π_{-1} is essentially independent of the chosen datum. More precisely, regularity is independent of the datum, and extra regularity is independent provided one restricts to data where ϕ_i are trivial on G^i_{sc}. I call such data *normalized*.

2.5.2 Howe factorization

Let (S, θ) be a pair consisting of a tame maximal torus $S \subset G$ and a character $\theta : S \to \mathbb{C}^\times$. Assume that p does not divide the order of the Weyl group.

Proposition 2.5.2. *There exists a twisted Levi tower $S = G^{-1} \subset G^0 \subset \cdots \subset G^d = G$ and a sequence of characters $(\phi_{-1}, \dots, \phi_d)$, such that ϕ_i is G^{i+1}-generic for all $i = 0, \dots, d-1$, $\phi_i|_{G^i_{\text{sc}}} = 1$, and $\theta = \prod_{i=-1}^d \phi_i|_S$.*

Remark 2.5.3. The groups in the tower are uniquely determined by θ, namely the root systems relative to S are the jumps of the filtration

$$R_r = \{\alpha \in R(S, G) | \theta \circ N_{E/F} \circ \alpha^\vee(E_r^\times)\}.$$

The characters ϕ_i are not uniquely determined, but any two data differ from each other by refactorization.

Definition 2.5.4. The pair (S, θ) is called tame regular *regular* (resp. *extra regular*) elliptic, if

1. S is an elliptic tame maximal torus;
2. the action of inertia on the root subsystem

$$R_{0+} = \{\alpha \in R(S,G) | \theta(N_{E/F}(\alpha^\vee(E_{0+}^\times))) = 1\}$$

   preserves a set of positive roots, where E/F is any tame Galois extension splitting S (note that R_{0+} is independent of the choice of E/F);
3. the character $\theta|_{S(F)_0}$ has trivial stabilizer for the action of $N(S,G^0)(F)/S(F)$ (resp. $\Omega(S,G^0)(F)$), where $G^0 \subset G$ is the reductive subgroup with maximal torus S and root system R_{0+}.

Theorem 2.5.5. *1. The Yu-datum obtained from a tame (extra) regular elliptic pair (S,θ) is a tame (extra) regular elliptic datum, whose refactorization class depends only on the pair.*

2. *The resulting supercuspidal representation $\pi_{(S,\theta)}$ is (extra) regular supercuspidal and depends only on the $G(F)$-conjugacy class of the pair.*
3. *The resulting map is a bijection from the set of $G(F)$-conjugacy class of tame (extra) regular elliptic pairs and the set of isomorphism classes of (extra) regular supercuspidal representations.*

2.5.3 The character formula

In the papers [Spi18] and [Spi21], Spice significantly expanded the computation of characters of [AS09] in order to accommodate all of Yu's representations. His formula mirrors the inductive nature of Yu's construction and bottoms out to the character of the depth-zero piece π_{-1}. If that piece itself is regular, which is by definition the case when π is regular, then the DeBacker–Reeder character formula can be absorbed into the argument. This formula was combined in [FKS21] with a refined version of the computation of the roots of unity described above in the toral case, leading to the following closed formula the character of $\pi_{(S,\theta)}$.

Theorem 2.5.6 (Spice). *Assume $p >> 0$, so that the exponential map converges on all topologically nilpotent elements in Lie$(G)(F)$. Let $\gamma \in G(F)$ be regular semisimple and let $\gamma = \gamma_s \cdot \gamma_u$ be a topological Jordan decomposition modulo center. Then $\Theta_{\pi_{(S,\theta)}}(\gamma)$ equals*

$$e(G)e(J)\epsilon_L(X^*(T_G)_\mathbb{C} - X^*(T_J)_\mathbb{C}, \Lambda) \sum_{\substack{g \in J(F)\backslash G(F)/S(F) \\ \gamma_s^g \in S(F)}} \Delta_{II}^{abs}[a,\chi''](\gamma_s^g) \cdot \theta(\gamma_s^g) \widehat{\iota}^J_{^gX}(\log(\gamma_u)) \tag{2.5}$$

2.6 Shallow values and comparison with real discrete series

It is worth recording the following consequence of Theorem 2.5.6 in the special case that $\gamma = \gamma_s$. In fact, this special case is valid under much less stringent restrictions on the base field F.

Corollary 2.6.1. *Let $\gamma \in G(F)$ be regular semi-simple and topologically semi-simple modulo center. Then $\Theta_{\pi_{(S,\theta)}}(\gamma)$ equals zero unless γ is (conjugate to) an element of $S(F)$, in which case it equals*

$$e(G)\epsilon_L(X^*(T_G)_\mathbb{C} - X^*(S)_\mathbb{C}, \Lambda) \sum_{w \in N(S,G)(F)/S(F)} \Delta_{II}^{abs}[a, \chi''](\gamma^w) \cdot \theta(\gamma^w) \quad (2.6)$$

An interesting feature of this formula is that (almost) all of its terms have natural interpretation over any local field, including $\mathbb{R}$! Indeed, the Kottwitz sign $e(G)$, the epsilon factor $\epsilon_L(X^*(T_G)_\mathbb{C} - X^*(S)_\mathbb{C}, \Lambda)$, the sum over the Weyl group, and the character θ, all make sense. Even the term Δ_{II}^{abs} makes sense abstractly, i.e. provided we specify the parameters a_α and χ_α.

The elements a_α were specified as $a_\alpha = \langle H_\alpha, X \rangle$, and $X \in \text{Lie}^*(S)(F)$ was an element that linearized the character θ. This seems like a strictly p-adic thing, but one can in fact rewrite the definition of a_α in a way that is more protable to the real world, namely as

$$\theta(N_{E/F}(\alpha(X+1)) = \Lambda(T_{E/F}(a_\alpha X)).$$

Strictly speaking this only specifies a_α in a certain coset, but that's enough. Now $X \mapsto X+1$ is to be interpteted as a truncated exponential. If we replace it with the true exponential over the real numbers, we arrive at the formula

$$\theta(N_{E/F}(\alpha(\exp(X))) = \Lambda(T_{E/F}(a_\alpha X)),$$

where now $F = \mathbb{R}$ and $E = \mathbb{C}$. This does specify a valid value for the parameter a_α.

The parameter χ''_α unfortunately does not seem to port easily from the p-adic to the real world. But this seeming deficiency will be resolved in the next section. For now, let us specify $\chi''_\alpha(z)$ to be the phase of the complex number z, i.e. $z/|z|$, when $\alpha > 0$, and the inverse when $\alpha < 0$. Then we obtain the following.

Corollary 2.6.2. *Let $\gamma \in S(\mathbb{R})$ be regular semi-simple. The Harish-Chandra character $\Theta_{\pi_{(S,\theta)}}(\gamma)$ of the discrete series representation equals*

$$e(G)\epsilon_L(X^*(T_G)_\mathbb{C} - X^*(S)_\mathbb{C}, \Lambda) \sum_{w \in N(S,G)(F)/S(F)} \Delta_{II}^{abs}[a, \chi''](\gamma^w) \cdot \theta(\gamma^w) \quad (2.7)$$

Thus, we see that the character values in the p-adic and real world are given by the same formula!

2.7 Covers

In the comparison between the p-adic and the real world in the previous subsection there was still one deficiency: The parameters χ''_α seemed somewhat auxiliary, and we found no way to align them in the two cases. But recall that there is a cleaner way to phrase the classification of real discrete series representations and their character formula, namely using the cover $S_\pm$. It turns out that this cover makes sense for any local field, and makes the character formula nicer in both settings, bringing the two worlds even closer together.

Let us first introduce the double cover. This is a construction that works for a torus S over a local field F equipped with a finite subset $R \subset X^*(S)$ that is stable under the action of $\Sigma = \Gamma \times \{\pm 1\}$ and does not contain 0. Given such a datum we introduce

$$S(F)_\pm = S(F)_R$$

to be the group consisting of elements $(s, (\delta_\alpha)_{\alpha\in R_{\text{sym}}})$, where $s \in S(F)$, $\delta_\alpha \in F_\alpha^\times$, $\sigma(\delta_\alpha) = \delta_{\sigma(\alpha)}$ for all $\sigma \in \Gamma$, and $\delta_\alpha/\delta_{-\alpha} = \alpha(s)$. Note that, δ_α can be replaced by $\eta_\alpha\delta_\alpha$ with $\eta_\alpha \in F_{\pm\alpha}$ provided we have $\eta_{\sigma\alpha} = \sigma(\eta_\alpha)$, and we regard the new element as equivalent if $\prod_{\alpha\in R_{\text{sym}}/\Gamma} \kappa_\alpha(\eta_\alpha) = 1$.

It is easy to see that projecting onto the element s gives a surjective map $S(F)_\pm \to S(F)$ whose kernel is the subgroup $\{\pm 1\}$ of $S(F)_R$, identified as the tuple $(1, (\eta_\alpha))$ for a collection η_α as above with $\prod_{\alpha\in R_{\text{sym}}/\Gamma} \kappa_\alpha(\eta_\alpha) = -1$. Thus we topologize $S(F)_\pm$ by lifting the topology on $S(F)$.

Example 2.7.1. Let us take S to be the 1-dimensional anisotropic torus over F that splits over a quadratic extension of F. Then $X^*(X) = \mathbb{Z}$ with the non-trivial element of $\Gamma_{E/F}$ acting by multiplication by -1 and $S(F) = E^1 = \ker(N_{E/F} : E^\times \to F^\times)$. Hilbert's theorem 90 asserts that the map $E^\times/F^\times \to E^1$ sending x to $x/\bar{x}$ is an isomorphism.

If we take $R = \{2, -2\}$ then the cover split canonically: its group of points consists of $\{(x, y)|x \in E^1, y \in E^\times/N(E^\times), x^2 = y/\bar{y}\}$ and the map $x \mapsto (x, x)$ is a splitting of the cover.

We take $R = \{1, -1\}$ then the cover has a group of points $\{(x, y)|x \in E^1, y \in E^\times/N(E^\times), x = y/\bar{y}\}$, which is isomorphic to $E^\times/N(E^\times)$ via projection to y. The kernel of this projection is $F^\times/N_{E/F}(E^\times) \cong \{\pm 1\}$. This cover may or may not split. We have the following cases:

1. $F = \mathbb{R}$. Then $S(F) = \mathbb{S}^1$, $S(F)_\pm = \mathbb{S}^1$, and the map $S(F)_\pm \to S(F)$ is the squaring map $\mathbb{S}^1 \to \mathbb{S}^1$. Thus, this cover never splits.

2. F non-archimedean and E/F unramified. Then the cover always splits canonically, with a retraction $E^\times/N(E^\times) \to \{\pm 1\}$ being given by the unramified quadratic character.

3. F non-archimdean and E/F tamely ramified and $q \equiv 1(4)$. Then the cover splits, but non-canonically. There are two natural retractions, namely the two tame quadratic characters $E^\times/N(E^\times)$ that extend the quadratic character of $k_E^\times/k_F^\times$, and the quotient of these retractions is the unramified quadratic character.

4. F non-archimedean and E/F tamely ramified and $q \equiv 3(4)$. Then the cover does not split.

Consider now a collection $a_\alpha \in E^\times$ for each $\alpha \in R$ s.t. $a_{\sigma\alpha} = \sigma(a_\alpha)$ and $a_{-\alpha} = -a_\alpha$ for all $\alpha \in R$ and $\sigma \in \Gamma$. Given such a set we obtain the function

$$a_S : S(F)_\pm \to \{\pm 1\} \qquad (\gamma, (\delta_\alpha)_\alpha) \mapsto \prod_{\substack{\alpha\in R_{\text{sym}}/\Gamma \\ \bar{\alpha}(\gamma)\neq 1}} \kappa_\alpha\left(\frac{\delta_\alpha - \delta_{-\alpha}}{a_\alpha}\right) \cdot \prod_{\substack{\alpha\in R_{\text{sym}}/\Gamma \\ \bar{\alpha}(\gamma)=1}} \kappa_\alpha(\delta_\alpha). \tag{2.8}$$

Both the numerator and the denominator of the argument of κ_α are non-zero elements of F_α of trace zero, so their quotient is a non-zero element of $F_{\pm\alpha}$. The values a_α for $\alpha \in R_{\text{asym}}$ are irrelevant to this function.

Fact 2.7.2. *The function a_S is genuine: For $\tilde{\gamma} \in S(F)_\pm$ and $\epsilon = -1 \in S(F)_\pm$ one has $a_S(\epsilon\tilde{\gamma}) = -a_S(\tilde{\gamma})$.*

We will apply this construction to the elliptic maximal torus $S \subset G$ and $R = R(S, G)$ being the absolute root system. When $F = \mathbb{R}$ one can see easily that half-sum of any set of positive roots integrates to a character $\rho : S(F)_\pm \to \mathbb{C}^\times$. In other words, $S(F)_\pm$ is the rho-cover that was used to give a more streamlined treatment of the classification of real discrete series and their character formula. We recall that these representations can be parameterized by $G(\mathbb{R})$-conjugacy classes of pairs $(S, \theta_\pm)$, where $S \subset G$ is an elliptic maximal torus, $\theta_\pm$ is a genuine character of $S(F)_\pm$ with regular differential, and that the character at a regular $s \in S(\mathbb{R})$ is given by

Let us now see what the role of the function a_S above is in this situation.

Lemma 2.7.3. *We have the equality*

$$a_S(\dot{\gamma}) = (-1)^{q(G)} e(G) \epsilon_L(1/2, X^*(T)_\mathbb{C} - X^*(S)_\mathbb{C}, \Lambda) \prod_{\langle \alpha, d\theta \rangle > 0} (\alpha^{1/2}(\gamma) - \alpha^{-1/2}(\gamma)).$$

Thus, this function captures the subtle behaviour of the Weyl discriminant – this discriminant is a root of unity of order 2 or 4, whose order matches exactly the order of the epsilon factor, but the oscilations are the subtle part, and this part is captured by the genuine function a_S. As a consequence, we have the following reformulation of the Harish-Chandra character formula:

Corollary 2.7.4. *The value of the Harish-Chandra character of the discrete series representation $\pi_{(S,\theta_\pm)}$ at a regular element $\gamma \in S(F)$ is given by*

$$e(G)\epsilon_L(1/2, X^*(T)_\mathbb{C} - X^*(S)_\mathbb{C}, \Lambda) \sum_{w \in N(S,G)(F)/S(F)} a_S(\dot{\gamma}^w)\theta_\pm(\dot{\gamma}^w).$$

Consider now a p-adic field F and a tame regular elliptic pair (S, θ). We showed how to construct the collection (χ''_α) from this pair. One can use this collection to produce a genuine character $\theta_\pm$ of $S(F)_\pm$. Thus, there is some mysterious passage from usual characters θ of $S(F)$ to genuine characters $\theta_\pm$ of $S(F)_\pm$, that mirrors the passage in the real case between $d\theta$ and $d\theta + \rho$.

Corollary 2.7.5. *In terms of the genuine character $\theta_\pm$, the formula for the Harish-Chandra character of $\pi_{(S,\theta)}$ at shallow elements $\gamma \in S(F)$ is given by*

$$e(G)\epsilon_L(1/2, X^*(T)_\mathbb{C} - X^*(S)_\mathbb{C}, \Lambda) \sum_{w \in N(S,G)(F)/S(F)} a_S(\dot{\gamma}^w)\theta_\pm(\dot{\gamma}^w).$$

We have finally arrived at a situation where the analogy between the real and p-adic cases is essentially perfect. Moreover, this analogy suggests that the function a_S in the p-adic case is some shadow of the complex phase of the Weyl discriminant, even through the latter doesn't have a complex phase, being a p-adic number.

This suggests further the following mystery: There should be a construction that produces from the pair $(S, \theta_\pm)$ directly the supercuspidal representation $\pi_{(S,\theta)}$, without first going through θ. This will hopefully clarify the nature of Yu's construction and the role of the FKS twisting.

We can also describe the character of the desired representation $\pi_{(S,\theta_\pm)}$ on all elements, by adapting Theorem 2.5.6, as follows.

Corollary 2.7.6. *Assume $p >> 0$, so that the exponential map converges on all topologically nilpotent elements in $Lie(G)(F)$. Let $\gamma \in G(F)$ be regular semi-simple and let $\gamma = \gamma_s \cdot \gamma_u$ be a topological Jordan decomposition modulo center. Then $\Theta_{\pi_{(S,\theta_\pm)}}(\gamma)$ equals*

$$e(G)e(J)\epsilon_L(X^*(T_G)_{\mathbb{C}} - X^*(T_J)_{\mathbb{C}}, \Lambda) \sum_{\substack{g \in J(F)\backslash G(F)/S(F) \\ \gamma_s^g \in S(F)}} [a_S \cdot \theta_\pm](\gamma_s^g)\widehat{\iota}^J_{g X}(\log(\gamma_u)) \tag{2.9}$$

2.8 The Kirillov and Gelfand-Graev models for SL_2 and PGL_2

We have described the construction of (at least regular) supercuspidal representations for all reductive groups under mild restrictions on F.

For the group $G = \mathrm{GL}_2$ and related groups, such as SL_2 or PGL_2, there are more classical constructions that look rather different.

The first construction is the Kirillov model. The underlying vector space is the space $\mathcal{C}_c^\infty(F)$ of all smooth compactly supported functions on the base field F, or rather a subspace of this vector space of very small codimension (0, 1, or 2) that contains $\mathcal{C}_c^\infty(F^\times)$. The model depends on the choice of a non-trivial character $\Lambda : F \to \mathbb{C}^\times$, which we fix. An element

$$g := \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

of the "mirabolic" subgroup operates on this vector space by sending a function f to the function

$$(gf)(x) = \Lambda(bx)f(ax).$$

This is true for *all* irreducible representations, i.e. the model so far is the same for all irreducible representations. Given an irreducible representation (π, V) of G, one can show that the subspace $V_0 \subset V$ consisting of those $v \in V$ for which[1]

$$\int_U \Lambda(u^{-1})\pi(u)vdu = 0.$$

Thus $V/V_0 \cong \mathbb{C}$ and choosing such an isomorphism, i.e. choosing a linear form $L : V \to \mathbb{C}$ with kernel V_0, gives the map

$$V \to \mathcal{C}_c^\infty(F), \qquad v \mapsto f_v, \qquad f_v(x) = L(\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} v),$$

which identifies (V, π) with a subspace of $\mathcal{C}_c^\infty(F)$ which can be shown to contain $\mathcal{C}_c^\infty(F^\times)$ with codimension 2 when π is principal series, codimension 1 when π is Steinberg, and codimension 0 when π is supercuspidal (i.e., when π is supercuspida, the Kirillov model equals $\mathcal{C}_c^\infty(F^\times)$).

The group G is generated by the mirabolic subgroup and the element

$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

[1] This integral needs to be regularized, as the limit of integrals over an increasing tower of open compact subgroups

Thus, to describe the Kirillov model fully, we only need to specify how w acts. This is the *only* place where the model differs for different representations. One can show that, when (π, V) is cuspidal, and we take the basis of $\mathcal{C}_c^\infty(F^\times)$ given by

$$\xi_{\chi,k}(x) = \begin{cases} \chi(x), & \mathrm{val}(x) = k \\ 0, & \text{else} \end{cases}$$

as χ runs over the characters of $F^\times$ and k runs over the integers, we have

$$\pi(w)\xi_{\chi,k} = \epsilon(\chi^{-1} \otimes \pi, 1/2, \Lambda) \cdot \xi_{\chi^{-1}\omega_\pi, -n(\chi^{-1}\otimes\pi,\Lambda)-k},$$

see [BH06, §37.3]; here $\epsilon(\pi, s, \Lambda)$ is the Godement–Jacquet local constant, whose absolute value is somewhat easier to understand, being a power of q governed by an integer $n(\pi, \Lambda)$, and whose complex phase equals $\epsilon(\pi, s, \Lambda)$ and is subtle, see [BH06, §24.3].

When one restricts the Kirillov model of a supercuspidal representation to $G' = \mathrm{SL}_2$, it breaks up generally into two pieces – the functions supported on $N(E^\times) \subset F^\times$ for a suitable quadratic extension E/F, and those supported on the complement. The two representations make up an L-packet, which corresponds to a stable class of tame regular elliptic pairs (S, θ), and the two representations correspond to the two rational classes in the stable class of the pair (S, θ). In the unique non-regular case, the restriction of the Kirillov model to G' breaks up into four pieces, by considering the functions on $F^\times$ supported on the subset

$$\{a \in F^\times | \kappa_{E/F}(a) = \epsilon_E \forall [E : F] = 2\},$$

where $\epsilon_E \in \{\pm 1\}$ and we have the condition $\epsilon_{E_1}\epsilon_{E_2} = \epsilon_{E_3}$ for the three quadratic extensions of F.

One can also consider the Kirillov model for $\bar{G} = \mathrm{PGL}_2$, simply by taking those representations of G with trivial central character. But here something rather intresting happens. Namely, one can interpret the Kirillov model as an instance of similitude theta correspondence. When one works out the details, one sees that the object being transferred by this correspondence is a genuine character of the double cover of the compact torus. Thus, in the special case of PGL_2, the mysterious machine we expect that produces representations from genuine characters of double covers of tori is furnished by the Kirillov model, i.e. the theta correspondence. This will be examined in a project.

There is yet another model for the irreducible representations of $G = \mathrm{GL}_2$ and $G' = \mathrm{SL}_2$, introduced in [GGPS16, Chapter 2,§4]. One starts with a quadratic extension E/F and a character $\theta : E^\times \to \mathbb{C}^\times$ and considers the space of square-integrable functions on F on which

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

operates by the formula

$$gf(x) = \int_F K_\theta(g, x, y) f(y) dy,$$

where the kernel $K_\theta(g, x, y)$ is defined as

$$a_E c_E \frac{\kappa_{E/F}(b)}{|b|} \kappa_{E/F}(u) \Lambda(\frac{du + av}{b}) \int_{E^1} \Lambda(-\frac{1}{b}(ut + vt^{-1}))\theta(t) dt,$$

which make sense when $b \neq 0$. Here $a_E = 2(1+q^{-1})(1+|\tau|)^{-1}$ with τ either a unit or a uniformizer so that $E = F(\sqrt{\tau})$, and $c_E = \int_E \Lambda(z\bar{z})dz$.

If this representation is restricted to $G' = \mathrm{SL}_2(F)$, then it breaks up into two pieces, according to the support of f being in $N_{E/F}(E^\times) \subset F^\times$ or the complement.

Gelfand–Graev–Pietetski-Shapiro give in [GGPS16, Chapter 2,§5, no. 4] a formula for the character of the representation of G (or the sum of the representations of G') associated to the pair $(E/F, \theta)$. In fact, one has a very simple formula for the integral over all θ:

$$\int_\theta \Theta_{\pi_{E/F,\theta}}(g)\theta(t)^{-1}\theta = 2\frac{\kappa_{E/F}(\mathrm{tr}(g)-\mathrm{tr}(t))}{|\mathrm{tr}(g)-\mathrm{tr}(t)|_F}$$

From this one computes via Fourier inversion

$$\Theta_{\pi_{E/F,\theta}}(g) = \int_{E^1} 2\frac{\kappa_{E/F}(\mathrm{tr}(g)-\mathrm{tr}(t))}{|\mathrm{tr}(g)-\mathrm{tr}(t)|_F}\theta(t)dt.$$

The relationship between the Gelfand–Graev and Kirillov models, as well as their relationship to the Adler/Yu-construction, is not clear. Similarly, the relationship between the character formulas due to Gelfand–Graev and Adler–Spice, is not clear.

3 Stable characters and endoscopy

In this lecture we will discuss a variation of the notion of a character due to Langlands, called "stable character", which is intimately related to Langlands' ideas about harmonica analysis and representation theory. A stable character is a linear combination of irreducible characters that has a strong conjugation-invariance property. The theory of endoscopy allows one to express individual characters in terms of stable characters, and thereby reduces usual harmonic analysis to "stable harmonic analysis".

3.1 The dual group and the L-group

We review the L-group of a connected reductive group following [Vog93, §2].

Let F be a field. Assume first that F is separably closed. Let G be a connected reductive F-group. Given a Borel pair (T, B) of G one has the based root datum $\mathrm{brd}(T, B, G) = (X^*(T), \Delta, X_*(T), \Delta^\vee)$, where $\Delta \subset X^*(T)$ is the set of B-simple roots for the adjoint action of T on $\mathrm{Lie}(G)$, and $\Delta^\vee \subset X_*(T)$ are the corresponding coroots. For a second Borel pair (T', B'), there is a unique element of $T'(F)\backslash G(F)/T(F)$ that conjugates (T, B) to (T', B'). This element provides an isomorphism $\mathrm{brd}(T, B, G) \to \mathrm{brd}(T', B', G)$. This procedure leads to a system of based root data and isomorphisms, indexed by the set of Borel pairs of G. The limit of that system is the based root datum $\mathrm{brd}(G)$ of G.

One can formalize the notion of a based root datum: we refer the reader to [Spr09, §7.4] for the formal notion of a root datum, to which one has to add a set of simple roots to obtain the formal notion of a based root datum. Based root data can be placed into a category, in which all morphisms are isomorphisms, for the evident notion of isomorphism of based root data. The classification of connected reductive F-groups [Spr09, Theorem 9.6.2, Theorem 10.1.1] can be stated as saying that $G \mapsto \mathrm{brd}(G)$ is a full essentially surjective functor from the category of connected reductive F-groups and isomorphisms to the category of based root data and isomorphisms. Moreover, two morphisms lie in the same fiber of this functor if and only if they differ by an inner automorphism.

Consider now a general field F, let F^s a separable closure, $\Gamma = \mathrm{Gal}(F^s/F)$ the Galois group. Given a connected reductive F-group G, there is a natural action of Γ on the set of Borel pairs of G_{F^s}, and this leads to a natural action of Γ on $\mathrm{brd}(G_{F^s})$. We denote by $\mathrm{brd}(G)$ the based root datum $\mathrm{brd}(G_{F^s})$ equipped with this Γ-action. Given two connected reductive F-groups G_1, G_2, an isomorphism $\xi : G_{1,F^s} \to G_{2,F^s}$ is called an *inner twist*, if $\xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1}$ is an inner automorphism of G_{1,F^s} for all $\sigma \in \Gamma$. The two groups G_1, G_2 are then called inner forms of each other. The functor $G \mapsto \mathrm{brd}(G)$ from the category of connected reductive F-groups to the category of based root data over F and isomorphisms is again essentially surjective. It maps inner twists to isomorphisms, and two inner twists map to the same isomorphism if they differ by an inner automorphism. The fiber over a given based root datum over F consists of all reductive groups that are inner forms of each other.

Given a based root datum $(X, \Delta, Y, \Delta^\vee)$ over F, its dual $(Y, \Delta^\vee, X, \Delta)$ is also a based rood datum over F. If G is a connected reductive F-group with based root datum $(X, \Delta, Y, \Delta^\vee)$, its dual $\widehat{G}$ is the unique split connected reductive group defined over a chosen base field (we will work with $\mathbb{C}$) with based root datum $(Y, \Delta^\vee, X, \Delta)$. Thus, given a Borel pair $(\widehat{T}, \widehat{B})$ of $\widehat{G}$ and a Borel pair (T, B) of G_{F^s}, one is given an identification $X_*(\widehat{T}) = X^*(T)$ that identifies the Weyl chambers associated to $\widehat{B}$ and B.

To form the L-group, one chooses a pinning $(\widehat{T}, \widehat{B}, \{Y_\alpha\})$ of $\widehat{G}$. The group of automorphisms of $\widehat{G}$ that preserve this pinning is in natural isomorphism with the group of automorphisms of $\mathrm{brd}(\widehat{G})$, hence with that of $\mathrm{brd}(G)$. The Γ-action on $\mathrm{brd}(G)$ then lifts to an action on $\widehat{G}$ by algebraic automorphisms, and ${}^LG = \widehat{G} \rtimes \Gamma$.

When G is quasi-split, (T, B) is an F-Borel pair, and $(\widehat{T}, \widehat{B})$ is a Γ-stable Borel pair of $\widehat{G}$, then the identification $X_*(T) = X^*(\widehat{T})$ is Γ-equivariant.

3.2 L-packets and stable characters

The point of departure is the basic form of the local Langlands correspondence, which should be a finite-to-one map from the set of isomorphism classes of irreducible representations of $G(F)$ to the set of $\widehat{G}$-conjugacy classes of L-parameters, i.e. homomorphisms

$$\varphi : \mathcal{L}_F \to {}^LG$$

from the Langlands group of the local field F

$$\mathcal{L}_F = \begin{cases} W_F, & F/\mathbb{R}, \\ W_F \times \mathrm{SL}_2(\mathbb{C}), & F/\mathbb{Q}_p \end{cases}$$

that are continuous on W_F, algebraic on $\mathrm{SL}_2(\mathbb{C})$, respect the Jordan decomposition, and the maps to Γ_F.

The fibers of this map are called L-packets. They partition the set of irreducible admissible representations into a disjoint union of finite sets, called L-packets. These packets are supposed to satisfy various properties, among which are:

1. If one member of the packet is tempered, then all members of the packet are tempered.
2. If one member of the packet lies in the essential discrete series, then all members lie in the essential discrete series.

Restricting attention to tempered representations, an L-packet Π should have the following further property, called *atomic stability*:

3. There exist complex numbers $(c_\pi)_{\pi\in\Pi}$, such that the function $\sum c_\pi\Theta_\pi$ is non-zero and stably invariant, i.e. its values at two strongly regular semi-simple elements $g_1, g_2 \in G(F)$ agree if g_1 and g_2 are conjugate in $G(E)$ for some Galois extension E/F. Moreover, no proper subset of Π has this property.

It is elementary to see that this property immediately implies that all c_π are non-zero, and that $S\Theta_\Pi := \sum c_\pi\Theta_\pi$ is determined up to multiplication by a non-zero complex number.

Definition 3.2.1. The function $S\Theta_\Pi$ is called the *stable character* of the L-packet Π.

This definition can be seen as somewhat provisional, because it specifies $S\Theta_\Pi$ only up to multiplication by a scalar. We will see in the next section that the definition can be refined, at the expense of assuming further conjectures, to provide a normalized function.

Due to linear independence of characters the function $S\Theta_\Pi$ determines the set Π. Since tempered L-packets are supposed to be indexed by tempered L-parameters, and the local Langlands correspondence is determined in a simple way from the tempered case, this leads to the following approach to characterize the local Langlands correspondence.

- For each tempered L-parameter φ, give a formula for the stable character of the corresponding L-packet.

3.3 Internal structure of L-packets: quasi-split groups

A first step in the theory of endoscopy is to give an enumeration of the elements of each L-packet Π. As we shall see, this will in particular give a way to normalize $S\Theta_\Pi$.

We begin with the case of quasi-split groups G. Recall that this means that there exists an F-rational Borel subgroup $B \subset G$. Let U be the unipotent radical of B. A character $\psi : U(F) \to \mathbb{C}^\times$ is called *generic* if its stabilizer for the action of $B(F)/U(F)$ is equal to the isomorphic image of $Z_G(F)$. A pair (B, ψ), where B is an F-Borel subgroup and ψ is a geneirc character of $U(F)$, is called a

Whittaker datum. We consider two such equivalent if they are $G(F)$-conjugate. The set of equivalence clases of Whittaker data is a torsor under the group $\mathrm{cok}(G(F) \to G_{\mathrm{ad}}(F))$.

The following expectation on tempered L-packets, called Shahidi's strong tempered L-packet conjecture, is essential for the internal structure.

4. A tempered L-packet contains a unique generic representation for a fixed Whittaker datum.

This already allows us to normalize $S\Theta_\Pi$ when G is quasi-split: We require $c_\pi = 1$ when $\pi \in \Pi$ is generic. Note that the stability of $S\Theta_\pi$ implies in particular that this function is stable under the conjugation action of $G_{\mathrm{ad}}(F)$ on $G(F)$, which implies that the collection c_π is stable under this action. Thus, if we choose $c_\pi = 1$ for one generic representation in Π, then we have $c_\pi = 1$ for all generic representations in Π.

To give the internal structure of Π_φ when G is quasi-split, we introduce $S_\varphi := \mathrm{Cent}(\varphi, \widehat{G})$. This group contains $Z(\widehat{G})^\Gamma$ as a subgroup and we can form $\bar{S}_\varphi = S_\varphi / Z(\widehat{G})^\Gamma$ and $\mathcal{S}_\varphi = \pi_0(\bar{S}_\varphi)$. Then we have the following expectation.

5. Upon fixing a Whittaker datum $\mathfrak{w}$ there exists a bijection $\iota_\mathfrak{w} : \Pi_\varphi \to \mathrm{Irr}(\mathcal{S}_\varphi)$ sending the unique $\mathfrak{w}$-generic representation to the trivial representation.

3.4 The conjectural formula for supercuspidal stable characters

We have seen how standard conjecures about L-packets have led to a stable function $S\Theta_\Pi$ associated to any tempered L-packet. This function is well-posed when G is quasi-split, and still ambiguous up to multiplication by a complex scalar otherwise. We will see that this ambiguity can be resolved for any G, but since this involves some additional notation and build-up we will first discuss the previously raised point of characterizing the local Langands correspondence by determining $S\Theta_\Pi$ in terms of the L-parameter φ of Π. Let us thus write $S\Theta_\varphi$.

In the case of classical groups, Arthur [Art13] used this approach. He determined $S\Theta_\varphi$ by relating the classical group to a general linear group via twisted endoscopy, and used the established local correspondence for general linear groups. This approach is very specific to classical groups, and does not work in general.

Here we will discuss a different approach, more closely aligned with Harish-Chandra's work. We will give an explicit formula for the function $S\Theta_\varphi$ when φ is a *supercuspidal* parameter, i.e. a discrete parameter with trivial monodromy. But we'll have to make a few technical assumptions.

Let us first recall the following expectations about the local Langlands correspondence:

6. The packet Π_φ consists of tempered representations if and only if φ is tempered, i.e. $\varphi(W_F)$ has bounded projection to $\widehat{G}$.

7. The packet Π_φ consists of essentially discrete series representations if and only if φ is discrete, i.e. $\bar{S}_\varphi$ is finite.
8. The packet Π_φ consists of supercuspidal representations if and only if φ is supercuspidal, i.e. φ is discrete and trivial on $\mathrm{SL}_2(\mathbb{C}) \subset \mathcal{L}_F$.

Let $\varphi : W_F \to {}^L G$ be a supercuspidal parameter. The main assumptions we have to make in order to obtain the desired formula for $S\Theta_\varphi$ are

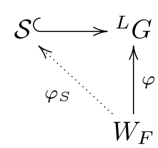
- G splits over a tame extension of F and p does not divide the order of the Weyl group of G.

Consider $\widehat{C} := \mathrm{Cent}(\varphi(I_F), \widehat{G})^\circ$. One can show that this is a torus in $\widehat{G}$ and that $\widehat{S} = \mathrm{Cent}(\widehat{C}, \widehat{G})$ is a maximal torus. By construction it is normalized by φ. Define

$$\mathcal{S} = \widehat{S} \cdot \varphi(\Gamma).$$

We note that this makes sense, even though φ is only defined on the dense subgroup $W_F \subset \Gamma$, because there is a finite index subgroup of W_F whose image under φ lies in $\widehat{S}$.

By construction we obtain the factorization



The group $\mathcal{S}$ is naturally an extension

$$1 \to \widehat{S} \to \mathcal{S} \to \Gamma \to 1,$$

and thus "looks like" the L-group of S. But here is an important wrinkle:

- *There is no natural isomorphism between $\mathcal{S}$ and ${}^L S$.*
- In fact, if we use the finite Galois group of the splitting field of S, then there may be no such isomorphism at all.

This is where the double cover $S(F)_\pm$ enters the picture. More precisely, the torus S whose dual is $\widehat{S}$ is naturally equipped with a stable class of embeddings $j : S \to G$, and hence with a subset $R(S, G) \subset X^*(S)$. So we can consider the double cover $S(F)_\pm$ associated to this subset.

One can show the following:

- In the general setting of a finite Σ-stable subset $R \subset X^*(S)$ one can associate an L-group ${}^L S^R_\pm$, which is generally a non-split extension of the Galois group $\Gamma_{E/F}$ of the splitting extension E/F of S by the dual torus $\widehat{S}$. One can further prove a local Langlands correspondence:

  $$\{\varphi : W_F \to {}^L S^R_\pm\}/\widehat{S} \leftrightarrow \mathrm{Hom}_{\text{cts,gen}}(S(F)^R_\pm, \mathbb{C}^\times),$$

  between L-parameters valued in ${}^L S^R_\pm$ and continuous genuine characters of $S(F)^R_\pm$.

- in the special case at hand, there is a *canonical* isomorphism ${}^L S^R_\pm \to \mathcal{S}$.

Therefore, we obtain, without any choices, a genuine character $\theta_\varphi : S(F)_\pm \to \mathbb{C}^\times$.

Our assumption $p \nmid W$ implies in particular $p \neq 2$, so the double cover splits over $S(F)_{0+}$. We can then consider the restriction $\theta_\pm|_{S(F)_{0+}}$ and linearize it by $X \in \mathrm{Lie}^*(S)(F)$, and set $a_\alpha = \langle H_\alpha, X\rangle$ as before. This gives us the function (2.8) and we can consider the following stable analog of the formula of Corollary 2.7.5:

- For strongly regular semi-simple and *shallow* (i.e. topologically semi-simple modulo center) element $\gamma \in G(F)$, set $S\Theta_\varphi(\gamma) = 0$ unless γ lies in the image of an admissible embedding $j : S \to G$. In that case, identify S with that image and set

$$S\Theta_\varphi(\gamma) = \epsilon_L(1/2, X^*(T)_\mathbb{C} - X^*(S)_\mathbb{C}, \Lambda) \sum_{w \in \Omega(S,G)(F)} a_S(\dot\gamma^w)\theta_\varphi(\dot\gamma^w).$$

We note that the choice of embedding j is immaterial, since the different possible j whose image contains γ are a torsor under $\Omega(S, G)(F)$.

We have thus given a formula for $S\Theta_\varphi$ on shallow elements. One can wonder if these elements are enough to pin down the representations in Π_φ. This was studied by Chan–Oi, who have shown that, while in general the answer is negative, there is a certain lower bound on $q = |k_F|$ which makes the answer positive.

In fact, if we assume that F has characteristic zero and p is sufficiently large, so that the exponential function converges on all topolocially nilpotent elements in the Lie algebra of G, then we can provide a formula for $S\Theta_\varphi(\gamma)$ on all strongly regular semi-simple elements $\gamma \in G(F)$, by following the lead of Corollary 2.7.6, as follows:

- Assume $p >> 0$, so that the exponential map converges on all topologically nilpotent elements in Lie$(G)(F)$. Let $\gamma \in G(F)$ be regular semi-simple and let $\gamma = \gamma_s \cdot \gamma_u$ be a topological Jordan decomposition modulo center. Define $S\Theta_\varphi(\gamma)$ as

$$e(J)\epsilon_L(X^*(T_G)_\mathbb{C} - X^*(T_J)_\mathbb{C}, \Lambda) \sum_{j : S \to J/\mathrm{st}} [a_S \cdot \theta_\varphi](\gamma_s^j)\widehat{S\iota_{j}}^J_X(\log(\gamma_u)) \quad (3.1)$$

  where J is the identity component of the centralizer of γ_s and j runs over the set of those admissible embeddings $j : S \to G$ that take image in J, taken up to J-stable conjugacy.

Thus, under the given assumptions of p or q, we have a unique characterization of the basic LLC for supercuspidal parameters. It was shown in [FKS21] that the constructions of [Kal19a] and [Kal19b] satisfy this characteriazation.

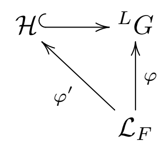
3.5 Endoscopic character identities: quasi-split groups

In the previous subsection we have given a formula for $S\Theta_\varphi$ for any supercuspidal parameter φ, subject to some conditions on F. We argued that this uniquely characterizes the basic local Langlands correspondence between representations and parameters. In this section we will discuss how to characterize the refined correspondence, i.e. the internal structure of the L-packets, and how to extract the Harish-Chandra characters of the individual representations from the stable characters.

For this, let us take any tempered parameter φ and let $s \in S_\varphi$ be a semi-simple element. Set $\widehat{H} = \text{Cent}(s, \widehat{G})^\circ$. By construction $\widehat{H}$ is normalized by φ and we can form

$$\mathcal{H} = \widehat{H} \cdot \varphi(\Gamma).$$

This is exactly parallel to the construction of $\mathcal{S}$. The same argument as before shows that $\mathcal{H}$ makes sense. We have the factorization



The group $\mathcal{H}$ fits into an extension

$$1 \to \widehat{H} \to \mathcal{H} \to \Gamma \to 1,$$

which in turn leads to a homomorphism

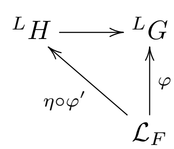
$$\Gamma \to \text{Out}(\widehat{H}).$$

If we take $H_{\bar{F}}$ to be the uniqe connected reductive group whose dual is $\widehat{H}$, then we have the identification $\text{Out}(H_{\bar{F}}) = \text{Out}(\widehat{H})$ and the above homomorphism gives a quasi-split F structure on $H_{\bar{F}}$, which we call H.

Definition 3.5.1. The group H is called the *endoscopic* group associated to (φ, s).

For a moment let us assume there exists, and fix, an isomorphism $\eta : \mathcal{H} \to {}^LH$.

The above diagram then becomes



and $\eta \circ \varphi'$ is a tempered parameter for H, hence there is an associated stable character $S\Theta_{\eta\circ\varphi'}$. The following expectation about tempered L-packets is called *endoscopic character identities*

9. Define the virtual character $\Theta^{\mathfrak{w}}_{\varphi,s} := \sum_{\pi\in\Pi_\varphi} \iota_{\mathfrak{w}}(\pi)(s) \cdot \Theta_\pi$. Then for any strongly regular semi-simple element $\delta \in G(F)$ we have

$$\Theta^{\mathfrak{w}}_{\varphi,s}(\delta) = \sum_\gamma \Delta[\mathfrak{w}, \eta](\gamma, \delta) \cdot S\Theta_{\eta\circ\varphi'}(\gamma), \tag{3.2}$$

where the sum runs over strongly regular semi-simple elements $\gamma \in H(F)$, taken up to stable conjugacy, and $\Delta[\mathfrak{w}, \eta](\gamma, \delta)$ is the Langlands–Shelstad transfer factor.

It is an elementary exercise to see that, for any $\pi \in \Pi_\varphi$, the Harish-Chandra character Θ_π is expressable as an explicit linear combination of the characters $\Theta^{\mathfrak{w}}_{\varphi,s}$, using Fourier theory on the finite group $\pi_0(\mathcal{S}_\varphi)$, namely

$$\Theta_\pi = |\pi_0(\mathcal{S}_\varphi)|^{-1} \sum_{s \in \pi_0(\mathcal{S}_\varphi)} \iota_{\mathfrak{w}}(\pi)^\vee(s)\Theta^{\mathfrak{w}}_{\varphi,s}. \tag{3.3}$$

We thus see:

Corollary 3.5.2. *The map $\iota_{\mathfrak{w}} : \Pi_\varphi \to Irr(\mathcal{S}_\varphi)$ and the character of each member of Π_φ are uniquely and effectively determined by the collection of stable characters $S\Theta_{\varphi'}$, for all endoscopic factorizations φ' of φ.*

The assumption of the existence of an isomorphism ${}^LH \to \mathcal{H}$ is generally not fulfilled. If we drop it, there are two ways to proceed. One way is to choose arbitrarily an extension $H_1 \to H$ whose kernel is an induced torus and such that $H_{1,\text{der}}$ is simply connected. This is called a z-extension. It always exists, but there is no natural choice for it. Furthermore, there always exists an embedding $\eta_1 : \mathcal{H} \to {}^LH_1$, but again there is no natural choice for it. The pair (H_1, ξ_1) is called a z-pair. Once it is chosen, one obtains from φ' the parameter $\varphi_1 = \eta_1 \circ \varphi'$ for H_1, as well as a transfer factor $\Delta[\mathfrak{w}, \eta_1] : H_1(F)_{\text{sr}} \times G(F)_{\text{sr}} \to \mathbb{C}$. One can now state the character identity.

There is an alternative approach, which does not involve fixing a z-pair, and is again based on double covers. It goes as follows.

- The minimal Levi subgroup $T^H \subset H$ embeds as a maximal torus in G, which provides subsets $R(T^H, H) \subset R(T^H, G) \subset X^*(T^H)$.
- The double cover of $T^H(F)$ coming from the finite set $R(T^H, G) \smallsetminus R(T^H, H)$ extends to a double cover $H(F)_\pm$ of $H(F)$.
- There is an associated L-group ${}^LH_\pm$, again an extension of Γ by $\widehat{H}$.
- Assuming LLC for groups closely related to H one can prove LLC for genuine representations of $H(F)_\pm$ and parameters valued in ${}^LH_\pm$.
- There is a *canonical* isomorphism $\mathcal{H} \to {}^LH_\pm$, so the parameter φ' naturally leads to a genuine L-packet $\Pi_{\varphi'}$.
- One can define a transfer factor $\Delta[\mathfrak{w}] : H(F)^{\text{sr}}_\pm \times G(F)^{\text{sr}} \to \mathbb{C}$. The definition is simpler than in the classical case and involves no auxiliary data.

One can now state the endoscopic character identities, without choosing further auxiliary data, as

$$\Theta^{\mathfrak{w}}_{\varphi,s}(\delta) = \sum_{\gamma \in H(F)^{\text{sr}}/\text{st}} \Delta[\mathfrak{w}](\dot\gamma, \delta) \cdot S\Theta_{\varphi'}(\dot\gamma).$$

3.6 Internal structure and character identities: non-quasi-split groups

Consider now a connected reductive F-group G that is not quasi-split. We still have to give a normalization of the function $S\Theta_\varphi$, and to state the endoscopic character identities. For both tasks the existence and uniqueness of generic representations was essential when G is quasi-split, and is missing for general G.

We follow ideas of Adams–Barbasch–Vogan [ABV92], [Vog93], and Kottwitz. The central insight is that one should not treat individual groups, but rather entire inner classes of groups together. Each inner class contains a unique quasi-split group, which serves as an organizing beacon. If we denote it by G^*, then the groups G in the inner class of G^* are those for which there exists an isomorphism $\xi : G^*_{\bar{F}} \to G_{\bar{F}}$ such that $\xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1}$ is an inner automorphism of $G^*_{\bar{F}}$. Such ξ is called an inner twist. It gives an identification ${}^L G^* \cong {}^L G$, so we can use ${}^L G^*$ as the recepient for L-parameters for the entire inner class. To lighten notation we shall write $\widehat{G}$ and ${}^L G$ in place of $\widehat{G}^*$ and ${}^L G^*$.

The first attempt is then to study tuples (G, ξ, π), where $\xi : G^* \to G$ is an inner twist and π is an irreducible representation of $G(F)$. These tuples should be taken up to a notion of isomorphism (for example, if we compose ξ with $\mathrm{Ad}(g)$ for $g \in G(F)$ then the new tuple should be seen as isomorphic to the old tuple). But it turns out that the pair (G, ξ) – called an *inner twist* – has too many automorphisms, namely the conjugation action of $g \in G_{\mathrm{ad}}(F)$ on $G(F)$ is an automorphism of the pair (G, ξ), but this automorphism can permute irreducible representations non-trivially, see [Vog93, §2]. Let us call the automorphisms of (G, ξ) that are of the form $\mathrm{Ad}(g)$ with $g \in G(F)$ *inner*. These automorphisms do not move representations of $G(F)$. Let us call the quotient of the group of all automorphisms by the group of inner automorphisms the group of *outer* automorphisms. So $\mathrm{Out}(G, \xi) = \mathrm{cok}(G(F) \to G_{\mathrm{ad}}(F))$, which is often non-trivial.

The idea of [ABV92, §2] is to enrich the tuple (G, ξ, π) with a further piece so as to eliminate outer automorphisms, i.e. consider tuples (G, ξ, z, π) up to a natural notion of isomorphism, such that the automorphisms of (G, ξ, z) are only given by $\mathrm{Ad}(g)$ for $g \in G(F)$. One simple way to achieve this is to take $z \in Z^1(\Gamma, G^*)$ with $\xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1} = \mathrm{Ad}(z_\sigma)$. The tuple (G, ξ, z) is called a *pure inner twist* and it has no outer automorphisms. The problem is however that not every (G, ξ) can be augmented with z, i.e. sometimes such a z does not exist.

A solution to this problem is to replace the cohomology $H^1(\Gamma, G^*)$ of the Galois group with the cohomology $H^1_{\mathrm{bas}}(\mathcal{E}, G^*)$ of a different group $\mathcal{E}$, with the following constraints

- There should be an injection $H^1(\Gamma, G^*) \to H^1_{\mathrm{bas}}(\mathcal{E}, G^*)$ and a surjection $H^1_{\mathrm{bas}}(\mathcal{E}, G^*) \to H^1(\Gamma, G^*_{\mathrm{ad}})$.
- There should be an interpretation of $H^1_{\mathrm{bas}}(\mathcal{E}, G^*)$ in terms of $\widehat{G}$, suitably functorial in G^*.

This can be achieved by considering extensions

$$1 \to u \to \mathcal{E} \to \Gamma \to 1,$$

where u is a (pro)algebraic F-group. The injectivity of $H^1(\Gamma, G^*) \to H^1_{\mathrm{bas}}(\mathcal{E}, G^*)$ is automatic, but the other requirements are subtle. In [Kal16] we take

$$u = \varprojlim_{E/F, n} \mathrm{Res}_{E/F} \mu_n / \mu_n,$$

and show that $H^2(\Gamma, u) = \widehat{\mathbb{Z}}$ and $H^1(\Gamma, u) = 0$, so there exists a distinguished isomorphism class of extensions $\mathcal{E}$ and it has no outer automorphisms, so giving the isomorphism class is as good as giving the extension itself. Then the other requirements above are met, and in fact we have

$$H^1_{\text{bas}}(\mathcal{E}, G^*) = \pi_0(Z(\widehat{\widehat{G}})^+)^*, \tag{3.4}$$

where $\widehat{\widehat{G}}$ is the universal cover of $\widehat{G}$ as a complex Lie group, and $Z(\widehat{\widehat{G}})^+$ is the preimage of $Z(\widehat{G})^\Gamma$ in $\widehat{\widehat{G}}$.

Using the surjectivity of $H^1_{\text{bas}}(\mathcal{E}, G^*) \to H^1(\Gamma, G^*_{\text{ad}})$ we can always find $z \in Z^1_{\text{bas}}(\mathcal{E}, G^*)$ such that $z \in Z^1(\Gamma, G^*)$ with $\xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1} = \text{Ad}(\bar{z}_\sigma)$, where $\bar{z} \in Z^1(\Gamma, G^*_{\text{ad}})$ is the image of z. The tuple (G, ξ, z) has no outer automorphisms. We can then let Π_φ be the set of isomorphism classes of tuples (G, ξ, z, π), with $\pi \in \Pi_\varphi(G)$. The expectation about internal structure and character identities formulated in [Kal16, §5.4] is then the following

10. For a fixed Whittaker datum $\mathfrak{w}$ on G^* there exists a (necessarily unique) bijection $\iota_{\mathfrak{w}} : \Pi_\varphi \to \text{Irr}(\pi_0(S^+_\varphi))$ that fits into the commutative diagram

$$\begin{array}{ccc} \Pi_\varphi & \xrightarrow{\iota_{\mathfrak{w}}} & \text{Irr}(\pi_0(S^+_\varphi)) \\ \downarrow & & \downarrow \\ H^1_{\text{bas}}(\mathcal{E}, G^*) & \longrightarrow & \pi_0(Z(\widehat{\widehat{G}})^+)^* \end{array},$$

where S^+_φ is the preimage of S_φ in $\widehat{\widehat{G}}$, the left map sends the isomorphism class of (G, ξ, z, π) to the cohomology class of z, the right map assigns central character, and the bottom map is (3.4), and such that $\iota_{\mathfrak{w}}$ sends the unique $\mathfrak{w}$-generic member $(G, 1, 1, \pi_{\mathfrak{w}})$ to the trivial representation, and for each semi-simple $\dot{s} \in S^+_\varphi$ satisfies the character identities

$$\Theta^{\mathfrak{w},z}_{\varphi,\dot{s}}(\delta) = \sum_{\gamma \in H(F)^{\text{sr}}/\text{st}} \Delta[\mathfrak{w}, z](\dot{\gamma}, \delta) \cdot S\Theta_{\varphi'}(\dot{\gamma}),$$

where z stands for a tuple (G, ξ, z),

$$\Theta^{\mathfrak{w},z}_{\varphi,s} := e(G) \sum_{\pi \in \Pi_\varphi(G)} \iota_{\mathfrak{w}}((G, \xi, z, \pi))(\dot{s}) \cdot \Theta_\pi,$$

and $\Delta[\mathfrak{w}, z] : H(F)^{\text{sr}}_{\pm} \times G(F)^{\text{sr}} \to \mathbb{C}$ is the normalized transfer factor as in [Kal16, §5.3].

It is shown in [Kal16, §5.6] and [Kal22a], by reinterpreting work of Langlands and Shelstad, that this expectation is satisfied when $F = \mathbb{R}$.

Note that we now have a normalization for $S\Theta_\varphi$ for any group G: Fix an arbitrary tuple (G, ξ, z) and arbitrary Whittaker datum $\mathfrak{w}$ on G^* and set

$$S\Theta_\varphi := \Theta^{\mathfrak{w},z}_{\varphi,1}.$$

The dependence of $\Theta^{\mathfrak{w},z}_{\varphi,s}$ on $\mathfrak{w}$ and z is very simple, see e.g. [Kal22b, §2.3], and one sees easily that the definition of $S\Theta_\varphi$ does not depend on the choices of $\mathfrak{w}$ and z. It also does not depend on ξ, because ξ is used twice – once in the tuple (G, ξ, z) and once to identify the L-groups of G and G^* – and these two uses cancel each other out.

3.7 The spectral side of the stable trace formula

In this section we consider a connected, semi-simple, simply connected, and anisotropic group G over a number field F, and we will discuss how the local theory laid out in the preceding sections can be used to stabilize the spectral side of the trace formula. The assumptions on G make it so that the trace formula and its stabilization take the simplest possible form without becoming trivial.

We will first review the trace formula. The fact that G is anisotropic implies that the quotient $G(F)\backslash G(\mathbb{A})$ is compact. Therefore the space of square-integrable automorphic forms decomposes discretely

$$L^2(G(F)\backslash G(\mathbb{A}_F)) = \widehat{\bigoplus}_\pi m(\pi) \cdot \pi.$$

The sum runs over all irreducible admissible representations of $G(\mathbb{A}_F)$, and $m(\pi)$ are natural numbers, often zero. Those π for which $m(\pi) > 0$ are called *automorphic*, and $m(\pi)$ is called the *automorphic multiplicity* of π. The trace formula takes the form

$$\sum_\pi m(\pi) \mathrm{tr}\, \pi(f) = \sum_{\gamma \in [G(F)]} \tau(G_\gamma) O_\gamma(f), \tag{3.5}$$

where

$$O_\gamma(f) = \int_{G(\mathbb{A})/G_\gamma(\mathbb{A})} f(x\gamma x^{-1}) dx/dx_\gamma.$$

We have used the canonical Tamagawa measures on $G(\mathbb{A})$ and $G_\gamma(\mathbb{A})$ and $\tau(G_\gamma)$ is the Tamagawa number of G_γ.

Thus the spectral side of the trace formula consists of traces of automorphic representations weighted by the multiplicities of these representations, and the geometric side consists of orbital integrals at elliptic elements weighted by the Tamagawa numbers of their centralizers.

The "trace formula" is the identity (3.5), which expresses the trace of the operator $R(f)$ acting on $L^2(G(F)\backslash G(\mathbb{A}))$ in two different ways: one more immediate, as the sum of traces of f on the various irreducible constituents π, and one less immediate, namely the sum of orbital integrals. It can be interpreted as giving a *formula* that expresses the sum of *traces* in geometric terms.

The distributions whose equality is asserted in (3.5) are invariant, but not stably invariant. The stabilization of the trace formula is the process that converts these distributions into stable distributions, and expresses the original distributions in terms of these stable distributions.

Let us write TF^G_{spec} for the left-hand side, and TG^G_{geom} for the right-hand side. The stabilization process defines two new distributions, STF^G_{spec} and STF^G_{geom} that are stably invariant and asserts an identities

$$STF^G_{\text{geom}}(f) = STF^G_{\text{spec}}(f)$$

as well as

$$TF^G_*(f) = \sum_H \iota(G, H) STF^H_*(f^H). \tag{3.6}$$

In this section, we will discuss the second of these two identities in the case $* = \text{spec}$. In fact, in order to simplify the discussion a bit more, we will focus on the tempered part of $L^2(G(F)\backslash G(\mathbb{A}))$. Even for anisotropic groups, not every automorphic representation is tempered – for example the trivial representation is automorphic, but is not tempered. Let $TF^G_{\text{spec,temp}}$ be the part of TF^G_{spec} that involves only the tempered representations π.

The identity (3.6) is based, in addition to the local endoscopic character identities reviewed in the previous sections, also on on Arthur's global conjecture [Art89, Conjecture 8.1], which gives a decomposition

$$L^2_{\text{temp}}(G(F)\backslash G(\mathbb{A})) = \bigoplus_{\varphi}\bigoplus_{\pi} m(\psi,\pi)\pi.$$

The first sum is over $\widehat{G}$-conjugacy classes[2] of "global generic Arthur parameters", which are L-homomorphisms $\mathcal{L}_F \to {}^LG$ involving the hypothetical Langlands group $\mathcal{L}_F$ of the global field F that are continuous on $\mathcal{L}_F$ and do not factor through a proper Levi subgroup. The group $\mathcal{L}_F$ should come equipped with homomorphisms $\mathcal{L}_{F_v} \to \mathcal{L}_F$ from the Langlands groups of the localizations F_v of F at all places v.

Composing φ with such a homomorphisms provides a tempered L-parameter $\varphi_v : \mathcal{L}_{F_v} \to {}^LG$. The local conjectures reviewed in the previous sections predicts the existence of a local tempered L-packet $\Pi_{\varphi_v}(G)$, equipped with an injection $\iota_{\mathfrak{w}_v,\xi_v,z_v} : \Pi_{\varphi_v}(G) \to \text{Irr}(\pi_0(S^+_{\varphi_v}))$, which depends on realizing G as a rigid inner twist $(\xi_v, z_v) : G^*_v \to G_v$ of its quasi-split inner form G^*_v, and a choice of a Whittaker datum $\mathfrak{w}_v$ for G^*_v. Write $\langle\pi_v,\dot{s}\rangle_{\mathfrak{w}_v,\xi_v,z_v}$ for the value at $\dot{s} \in S^+_{\varphi_v}$ of the character of the irreducible representation of $\pi_0(S^+_{\varphi_v})$ associated by this map to $\pi_v \in \Pi_{\varphi_v}(G)$.

In the global context we shall fix a global realization $(\xi, z) : G^* \to G$ of G as a rigid inner twist of its quasi-split inner form G^*, as well as a global Whittaker datum $\mathfrak{w}$. Then $\mathfrak{w}_v, \xi_v, z_v$ will be obtained by localizing these global data. This will have the effect that $\langle\pi_v, -\rangle_{\mathfrak{w}_v,\xi_v,z_v} = 1$ at those places v where all the data are unramified.

The second sum runs over admissible adelic representations $\pi = \otimes'_v\pi_v$, where $\pi_v \in \Pi_{\varphi_v}(G)$ for all v. Then $\langle\pi_v, -\rangle = 1$ for almost all v and we can form $\langle\pi,\dot{s}\rangle = \prod_v\langle\pi_v,\dot{s}\rangle$ for $s \in S^+_\varphi$, where we have used the injection $S_\varphi \to S_{\varphi_v}$. One can show that $\langle\pi, z\rangle = 1$ for $z \in [Z(\widehat{\widehat{G}})]^+$, so $\langle\pi,-\rangle$ descends to $\mathcal{S}_\varphi = S^+_\varphi/[Z(\widehat{\widehat{G}})]^+ = S_\varphi/Z(\widehat{G})^\Gamma$, and does not depend on the global data $\mathfrak{w}$, ξ, and z, cf. [Kal18, Propositions 4.5.2]. The integer $m(\varphi,\pi)$ is then defined as (cf. [Art89, (8.5)][3])

$$m(\varphi,\pi) = \text{mult}(1, \langle\pi,-\rangle) = |\mathcal{S}_\varphi|^{-1}\sum_{s\in\mathcal{S}_\varphi}\langle\pi,s\rangle.$$

Remark 3.7.1. When the group $\mathcal{S}_\varphi$, as well as its local analogs $\pi_0(S_{\varphi_v})$, are abelian, which happens for example when G is a classical group, then the above sum is either 0 or 1, and it is 1 precisely when the character $\langle\pi,-\rangle : \mathcal{S}_\psi \to \mathbb{C}^\times$

[2]We are using again the Hasse principle here. When it doesn't hold, the notion of equivalence of global parameters is slightly more complicated, cf. [Kot84, §10.4].

[3]The superscript $+$ in S^+_ψ in loc. cit. has a different meaning from the superscript $+$ used here: in loc. cit. it refers to the possibility of G being a disconnected reductive group, while for us it refers to the universal cover of $\widehat{G}$ that is needed to treat general non-quasi-split groups.

is trivial. Therefore Arthur's conjecture can be stated in the following shorter form

$$L^2_{\text{temp}}(G(F)\backslash G(\mathbb{A})) = \bigoplus_{\varphi} \bigoplus_{\pi:\langle\pi,-\rangle=1} \pi.$$

Remark 3.7.2. Arthur's conjecture extends beyond the tempered case, but one has to replace local tempered L-packets with local A-packets and local tempered L-parameters with local A-paramters. In the formula for $m(\varphi,\pi)$, the trivial character of the global centralizer S_φ must be replaced by Arthur's character ϵ_ψ, which explicit but very subtle.

Assuming these conjectures, the tempered spectral side of (3.5) becomes

$$\sum_{\varphi}\sum_{\pi} |\mathcal{S}_\varphi|^{-1} \sum_{s\in\mathcal{S}_\varphi} \langle \pi, s\rangle \mathrm{tr}\,\pi(f), \tag{3.7}$$

where again φ runs over the set of $\widehat{G}$-conjugacy classes of discrete generic Arthur parameters, and π runs over all members of the global A-packet

$$\Pi_\varphi = \{\pi = \otimes'_v \pi_v \,|\, \pi_v \in \Pi_{\varphi_v},\ \langle \pi_v, -\rangle = 1 \text{ for almost all } v\}.$$

We use a factorizable test function $f = \prod_v f_v$ and switch the sums over s and π, to turn (3.7) into

$$\sum_{\varphi} |\mathcal{S}_\varphi|^{-1} \sum_{s\in\mathcal{S}_\varphi} \prod_v \sum_{\pi_v} \langle \pi_v, \dot{s}\rangle \mathrm{tr}\,\pi_v(f_v). \tag{3.8}$$

Recall the construction of an endoscopic datum from a pair (φ, s) in the local case. The same construction works in the global case, and in fact provides a correspondence

$$(\varphi, s) \leftrightarrow (H, s, \mathcal{H}, {}^L\xi, \varphi'). \tag{3.9}$$

On the left we have pairs consisting of a global generic Arthur parameter φ and an element $s \in S_\psi$, while on the right we have tuples consisting of an elliptic endoscopic datum $(H, s, \mathcal{H}, \xi)$, an extension[4] of ξ to an L-isomorphism ${}^L\xi : \mathcal{H} \to {}^LH$ and a generic Arthur parameter ψ' for H. The correspondence between both sides is obtained as follows. Given $(H, s, \mathcal{H}, {}^L\xi, \psi')$ we set $\psi = {}^L\xi \circ \psi'$.

The local endoscopic character identities are

$$\sum_{\pi_v} \langle \pi_v, \dot{s}\rangle \mathrm{tr}\,\pi_v(f_v) = S\Theta_{\varphi_v^H}(f_v^H).$$

With this, (3.8) becomes

$$\sum_H \iota(G,H) \sum_{\varphi^H} |\mathcal{S}_{\varphi^H}|^{-1} S\Theta_{\varphi^H}(f^H). \tag{3.10}$$

We have not discussed here how the quantity $\iota(G,H)|\mathcal{S}_{\varphi^H}|^{-1}$ arises from the quantity $|\mathcal{S}_\varphi|^{-1}$. The two are not equal, and their discrepancy accounts for the failure of the correspondence (3.9) to be bijective. We are also being vague about the equivalence up to which the parameters φ^H are to be taken, which as we have already mentioned is more subtle than $\widehat{H}$-conjugacy when H does

[4]We are using here the assumption that G_{der} is simply connected, which guarantees the existence of ${}^L\xi$, cf. [Lan79]

not satisfy the Hasse principle. Details can be found in [Kot84, §11], especially Proposition 11.2.1 there.

Identity (3.10) shows that we have identity (3.6) provided we make the following definition:

Definition 3.7.3.

$$STF^G_{\text{spec,temp}}(f) := \sum_{\varphi} |\mathcal{S}_\varphi|^{-1} S\Theta_\varphi(f),$$

as φ runs over the set of equivalence classes of global generic Arthur parameters.

This definition does indeed produce a stable distribution, and is in fact quite analogous to the defining equation $TF^G_{\text{spec,temp}}(f) = \sum_\pi m(\pi)\text{tr}\,\pi(f)$, where π runs over the tempered automorphic representations and $m(\pi)$ is the automorphic multiplicity of π.

4 PROJECTS

4.1 Gelfand–Graev Fourier transform

In this project we will relate two different stable character formulas for the group SL_2: the character formulas discussed in these lectures, and the formulas given in [GGPS16, Chapter 2,§5, no. 4]. The goal of this project is to establish a clear path between these formulas and explain how the various terms fit together.

The formula in applies even to the case $p = 2$. We will want to develop the Adler-Spice formula in that case as well.

1. Relate the representations constructed by Adler to those of GGPS when $p \neq 2$. Both are indexed by the same data.
2. Compare the character formulas at shallow elements.
3. Compare the character formulas at deep elements.
4. Combine the two comparisons.
5. Write out the construction of representations a-la Adler, but for $p = 2$.
6. Follow Adler–Spice to compute the character for $p = 2$.
7. Compare the formulas for $p = 2$.

4.2 Stable characters for $p = 2$

In this project we will examine the conjectural formula for the stable character of a supercuspidal L-packet. It is currently formulated in the setting of an odd prime, but there should be an extension to $p = 2$. In that setting one has to deal with double covers of wildly ramified tori, which are not as well understood.

Thus, part of the project will be to obtain a better understsanding of the covers of wildly ramified tori.

We will begin with the groups SL_2 and PGL_2 over $\mathbb{Q}_2$ and will try to find a good formulation there, which can hopefully be extrapolated to the general case.

Here is an outline.

1. Examine $SL_2/\mathbb{Q}_2$ and extend the conjectural formula for the stable character, using the double covers, to that case.
   (a) Examine the splitting behavior of the cover, including the possible χ-data and a-data.
   (b) Find a clean way to extract a-data when $p = 2$, i.e. the element $\langle H_\alpha, X\rangle$. Try to use the fact that the exponential map $\mathrm{Lie}(S)(F) \to S(F)$ factors through the double cover $S(F)_\pm \to S(F)$ and mimic the real case.
2. Consider the case of a general group. Here we will use for each root the insights from the SL_2-case.

4.3 The Kirillov model in terms of Yu data

There is a classic construction of representations of $\mathrm{GL}_2(F)$ for a non-archimedean local field F, via the so-called Kirillov model, as explained for example in [BH06, §§36-40]. In modern terms it can be expressed as a similitude theta-correspondence, namely between $\mathrm{GL}_2 = \mathrm{GSp}_2$ and $\mathrm{GO}(V)$, where V is the 2-dimensional F-vector space E, seen as quadratic space via the norm map $N : E \to F$. The similitude character on $\mathrm{GL}_2(F)$ is the determinant; we will denote by λ the similitude character of $\mathrm{GO}(V)$. The group

$$\{(g, h) \in \mathrm{GL}_2(F) \times \mathrm{GO}(V) | \det(g)\lambda(h) = 1\}$$

embeds into $\mathrm{Sp}_4(F)$, and this embedding lifts to $\mathrm{Mp}_4(F)$. The Weil representation on $\mathrm{Mp}_4(F)$, restricted to $\mathcal{G}$, provides the desired theta-correspondence, as explained in the paper of Brooks Roberts [Rob96].

The group $\mathrm{GO}(V)$ contains $E^\times$ (acting by multiplication on $E = V$) as a subgroup of index 2 and the representations of $\mathrm{GO}(V)$ we are interested in transferring are obtained as irreducible inductions from regular characters of $E^\times$. The similitude character λ on $\mathrm{GO}(V)$ restricts to $E^\times$ to the norm map $N : E^\times \to F^\times$. Therefore we can use the slightly smaller group for similitude theta-correspondence

$$\mathcal{G} = \{(g, e) \in \mathrm{GL}_2(F) \times E^\times | \det(g)N(e) = 1\}.$$

This is the group that appears in [BH06, §38]. It is shown in [BH06, 39.2.3] that the representation π_θ of GL_2 corresponding to a character $\theta : E^\times \to \mathbb{C}^\times$ descends to PGL_2 if and only if θ descends to a genuine character of the cover $E^\times/N(E^\times)$ of $E^\times/F^\times$.

The goal of this project is to express the representation π_θ on $\mathrm{PGL}_2(F)$ in terms of Yu's construction, i.e. write down explicitly the Yu-datum of this representation. For this one can use the paper [LM18], where they do this work in a fair bit of generality, but not in the similitude case. So some adaptation will be necessary.

4.4 Character formulas for limits of discrete series of p-adic $\mathbf{SL}_2$

Consider the group $G = \mathrm{SL}_2(F)$ for a non-archimedean local field F of characteristic zero, and the representation $i_B^G(\chi)$ for a non-trivial $\chi : F^\times \to \{\pm 1\}$. This representation is unitary but reducible, and decomposes as a direct sum $\pi^+ \oplus \pi^-$. The labeling is not unique. If we fix a Whittaker datum we can take π^+ to be the unique generic constitutent.

The goal of the project is to compute the characters of $\pi^\pm$. It would be interesting to see if the results bare the same phenomenon as in the case of $F = \mathbb{R}$, namely that the character formulas of these representations are parallel to those of discrete series (in this case supercuspidal) representations, i.e. that they behave as "limits of discrete series".

As remarked in §1.11, it is enough to compute $\Theta_{\pi^+} - \Theta_{\pi^-}$. For this, one can use (3.3) and (3.2). This will involve computing the refined L-parameters of the representations $\pi^\pm$ and their endoscopic transfers.

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