Basic LLC:

$$\mathcal{L}_F := \begin{cases} W_F, & F/\mathbb{R} \\ W_F \times SL_2(\mathbb{C}), & F/\mathbb{Q}_p \end{cases}$$

<u>Conj</u> (Basic LLC): There is a finite-to-1 map

$$\operatorname{Irr}(G) \to \Phi(G) = \{\mathcal{L}_F \xrightarrow{\varphi} {}^L G\} / \hat{G}\text{-conj}$$

$\Pi_\varphi := LLC^{-1}(\varphi)$ L-packets

<u>Expect</u>:

$\exists \pi \in \Pi_\varphi(G)$ temp $\Leftrightarrow$ $\forall \pi \in \Pi_\varphi(G)$ temp
e.d.s. e.d.s.

$\Leftrightarrow \varphi$ temp
disc

Atomic stability:

<u>Def</u>: $f : G_{sr} \to \mathbb{C}$ is <u>stable</u> if $f(g) = f(g')$ whenever g, g' are conj $G(\bar{F})$

Recall: F any local field of char 0 (1)

- $\left\{ \begin{matrix} \text{irred reg d.s.} \\ \text{of } G \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \{(S,\theta)\}/G\text{-conj}$

$S \subset G$ ell max torus

$\theta_1 : S_1 \to \mathbb{C}^\times$ reg gen char

$$\Theta_{\pi(S,\theta_1)}(\underset{\underset{S}{\cap}}{\gamma}) = e(G)\,\varepsilon(T_G - T_S) \sum_{w} [a_S \cdot \theta_1](\gamma^w)$$

p'- ~~prime~~ order

(uniquely determines (H.C / $\mathbb{R}$)
(Chan-Oi) if S unram, $q \gg 0$)

Goal:
- introduce LLC
- unique char of LLC
- refined LLC
- global application

<u>Conj</u>: Assume φ tempered. There exists $S\Theta_\varphi := \sum_{\pi \in \Pi_\varphi(G)} z_\pi \cdot \Theta_\pi$, $z_\pi \in \mathbb{C}$

which is non-zero and stable.
No proper subset of $\Pi_\varphi(G)$ has this property.

<u>Fact</u>: $S\Theta_\varphi$ is unique up to rescaling, and $z_\pi \neq 0$.

<u>Observe</u>: $\Pi_\varphi(G)$ and $\mathbb{C}^\times \cdot S\Theta_\varphi$ determine each other.

<u>Unique char of Basic LLC</u>:

Enough to specify $S\Theta_\varphi$ in terms of φ.

- Arthur: Classical groups -> twisted endoscopy
- Today: explicit formula for φ is supercuspidal, $p \nmid \# W$. G tame

Def: φ supercuspidal $\Leftrightarrow$ φ discrete
$\varphi|_{SL_2} = 1$.

Assume $p \nmid \#W$, G tame.

Cor: $Z_{\hat{G}}(Z_{\hat{G}}(\varphi(I_F)^\circ)) =: \hat{S}$
max torus of $\hat{G}$ normalized by φ.

Set $\mathcal{S} := \hat{S} \cdot \varphi(W_F) \subseteq {}^L G$.

Fact: ·) $W_F \xrightarrow{\varphi} \mathcal{S} \subseteq {}^L G$

·) $1 \to \hat{S} \to \mathcal{S} \to \Gamma \to 1$

Magic: There is a canonical iso
$\mathcal{S} \xrightarrow{\sim} {}^L S_{\pm}$.

$\xrightarrow{LLC}$ $\theta_{\pm} : S_{\pm} \to \mathbb{C}^\times$ gen char.

$S \xhookrightarrow{j} G$ adm embeddings

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Conj: $S\Theta_\varphi(\gamma) = e(G)\,\varepsilon(T-S) \sum_{w \in (N_G(S)/S)(F)} [a_S \cdot \theta_x](\gamma^w)$

$\hat{S}$ p'-order

More generally

$$\boxed{S\Theta_\varphi(\gamma) = e(G)\,e(J)\,\varepsilon(T_G - T_J) \sum_{j: S \to J/\text{st}} [a_S \cdot \theta](\gamma_0^j)\, S\{\hat{J}\}_{jX}(\log \gamma_u) \quad \gamma_0 \text{ top ss} \quad J = Z_G(\gamma_s)^\circ.}$$

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Internal structure of $\Pi_\varphi(G)$:

Assume G is q-split: $\exists B \subset G$ Borel sub over F.

Conj: $\exists$ Bij $\Pi_\varphi(G) \overset{i_4}{\longleftrightarrow} \mathrm{Irr}(\pi_0(S_\varphi/Z_{\hat{G}}^{\Gamma}))$

$S_\varphi = Z_{\hat{G}}(\varphi)$

Q: who corresp. to $1 \in$ RHS?

Conj (Shahidi): Fix a generic

$\psi : U \to \mathbb{C}^\times \quad (B = TU)$.

$\exists!\ \pi \in \Pi_\varphi(G)$ ~~which~~ which is

ψ-generic, i.e. $\mathrm{Hom}_U(\pi, \mathbb{C}_\psi) \neq 0$

Q: How unique is π_ψ?

Endoscopic character identities:

Let $s \in S_\varphi$ s.s. Def $\hat{H} := Z_{\hat{G}}(s)^\circ \subset \hat{G}$.

$\mathcal{H} = \hat{H} \cdot \varphi(W_F) \subset {}^L G$. Then

·) $\varphi : L_F \to \mathcal{H} \hookrightarrow {}^L G$

·) $1 \to \hat{H} \to \mathcal{H} \to \Gamma \to 1$

Magic': There is a double cover

$H_2 \to H$ and a can iso

$\mathcal{H} \to {}^L H_2$.

<u>Thus</u>: $\varphi' : L_F \to {}^L H_{\pm} \rightsquigarrow \Pi_{\varphi'}(H_{\pm})$.

<u>Conj</u>: $$S\Theta_{\varphi'}(f') = \underbrace{\sum_{\pi\in\Pi_\varphi(G)} \mathrm{tr}(\tau_\psi(\pi)(s))\cdot\Theta_\pi(f)}_{\Theta^{\psi}_{\varphi,s}}$$

$f' \leftrightarrow f$

$$\Updownarrow$$

$$\sum_{\gamma\in H/st} \Delta(\gamma,\delta)\, S\Theta_{\varphi'}(\gamma) = \Theta^{\psi}_{\varphi,s}(\delta)$$

$\delta \in G_{rs}$

<u>Fact</u>: the above Conj uniquely determines τ_ψ.

Stabilization of the aut spectrum:

Let $G/\mathbb{Q}$ conn red.

$$\mathbb{A} = \prod_{p \le \infty}' \mathbb{Q}_p$$

$$L^2_{\substack{disc\\temp}}(G(\mathbb{Q})\backslash G(\mathbb{A})) \curvearrowleft G(\mathbb{A}) = \prod_p' G(\mathbb{Q}_p)$$

$$\cup$$

$$\pi = \bigotimes_p \pi_p, \quad \pi_p \in \mathrm{Irr}(G(\mathbb{Q}_p))$$

$$L^2_{\substack{disc\\temp}}(G(\mathbb{Q})\backslash G(\mathbb{A})) = \widehat{\bigoplus_{\pi \in \mathrm{Irr}(G(\mathbb{A}))}} \pi^{m(\pi)}$$

Def: $m(\pi) > 0 \Rightarrow \pi$ automorphic

$m(\pi)$ is the aut multiplicity.

Q: Compute $m(\pi)$?

Conj (Kottwitz): For any $\pi \in \mathrm{Irr}(G(\mathbb{A}))$,

$$\pi = \bigotimes{}' \pi_p$$

$$m(\pi) = \sum_{\substack{\varphi : \mathcal{L}_{\mathbb{Q}} \to {}^L G \\ \pi_p \in \Pi_{\varphi_p}(G)}} \mathrm{mult}\Big(1, \bigotimes_p \eta_\varphi(\pi_p)\Big)$$

$$\begin{tikzcd} \mathcal{L}_{\mathbb{Q}} \\ \mathcal{L}_{\mathbb{Q}_p} \arrow[u, hook] \end{tikzcd}$$

$\mathcal{L}_{\mathbb{Q}}$ is the Langlands group of $\mathbb{Q}$

$$\begin{tikzcd} \varphi : \mathcal{L}_{\mathbb{Q}} \arrow[r] & {}^L G & S_\varphi = \mathrm{Cent}(\varphi, \hat{G}) \arrow[d, hook] \\ \varphi_p : \mathcal{L}_{\mathbb{Q}_p} \arrow[u] & & S_{\varphi_p} = \mathrm{Cent}(\varphi_p, \hat{G}) \end{tikzcd}$$

Q: How do you approach $m(\pi)$?

$$f \in C_c^\infty(G(\mathbb{A}))$$

$$\mathrm{Tr}(f \mid L^2) = \sum_{\pi} m(\pi) \cdot \Theta_\pi(f)$$

$$= \sum_{\pi} \sum_{\varphi} |S_\varphi|^{-1} \sum_{s \in S_\varphi} \prod_p \mathrm{tr}(\eta_\varphi(\pi_p)(s)) \cdot \Theta_{\pi_p}(f)$$

$$= \sum_{\varphi} \prod_p \underbrace{\sum_{\pi_p} \mathrm{tr}(\eta_\varphi(\pi_p)(s))\, \Theta_{\pi_p}(f)}$$

$$(\varphi, s) \leftrightarrow (H_1, s, \varphi')$$

$$= \sum_{\substack{(H_1, s) \\ \text{ell endo}}} \iota(G,H) \underbrace{\sum_{\varphi'} |S_{\varphi'}|^{-1}\, S\Theta_{\varphi'}(f')}_{ST(f', L^2([H]))}.$$