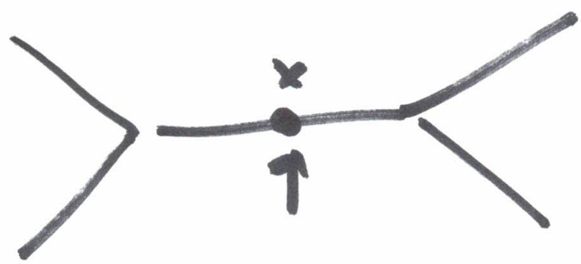
G p-adic, $\underline{G}(F) = G$

$\underline{G}$ conn red F-group

$x \in \mathcal{B}$, $G_x / G_{x,0+} \cong \overline{G}_x$ k-pts of alg k-grp

disc

$G = PGL_2$



Recall: • $F = \mathbb{R}$

$$\{\text{reg. d.s. reps}\} \longleftrightarrow \{(S,\theta)\}/G\text{-conj}$$

$S \subset G$ ell max torus

$\theta : S \to \mathbb{C}^\times$ reg char

$$\Theta_{\pi(S,\theta)}(\underset{\cup}{\gamma}) = (-1)^{q(G)} \sum_w \frac{\theta(\gamma^w)}{\prod_{\alpha>0}(1-\alpha(\gamma^w)^{-1})}$$

• $F/\mathbb{Q}_p$

$$\{\text{reg s.c. d.z}\} \longleftrightarrow \{(S,\theta)\}/G\text{-conj}$$

$S \subset G$ ell max torus, max unramified

$\theta : S \to \mathbb{C}^\times$ reg char, d.z

p'-order

$$\Theta_{\pi(S,\theta)}(\underset{\cup}{\gamma}) = (-1)^{r_G - r_S} \sum_w \theta(\gamma^w)$$

Note: $F = \mathbb{R}$, S need not exist, unique

$F/\mathbb{Q}_p$, S always exists, rarely unique

<u>Goal</u>: General depth reg s.c.

- Construction
- Parameterization
- Characters

<u>Yu's construction</u>:

$$\left\{ \begin{matrix} G^0 \subset \cdots \subset G^d = G \\ \phi_0, \quad \cdots \quad \phi_d \end{matrix} \right\} \xrightarrow{\text{Yu}} \left\{ \begin{matrix} \text{s.c. reps} \\ \text{of } G \end{matrix} \right\}$$

$$\pi_{-1}$$

Kim 2007, Fintzen 2021: Surj, when $p \nmid \#W$

Hakim–Murnaghan 2008: Fibers (refactorization)

<u>Def</u>: π regular $\Leftrightarrow$ π_{-1} regular

<u>Howe factorization</u>:

<u>Thm</u>: Let $S \subset G$ be a tame max torus. $\theta : S \to \mathbb{C}^\times$ a character, $p \nmid \#W$.

There exists $S \subset G^0 \subset \cdots G^d = G$, (2

ϕ $\phi_{-1}, \phi_0, \ldots, \phi_d$

characters, ϕ_i is G^{i+1}-generic for $i = 0, \ldots, d-1$, s.t. $\theta = \prod_{i=-1}^{d} \phi_i |_S$.

This datum is unique up to refactorization.

<u>Thm</u>: Bij $\{\text{reg. s.c. reps}\} \leftrightarrow$

$\{(S, \theta)\} / G\text{-conj}$

$S \subset G$ ell max torus

$\theta : S \to \mathbb{C}^\times$ reg char

$S \subset G^0$ is max unramified

<u>Char formula</u>:

Consider first the one-step case:

$(S \subset G, \phi_0 = \theta)$ generic char of depth $r > 0$

Adler-Spice proved:

$\mathfrak{S} = \mathbb{C}$

Theorem (Adler-Spice). *Let $\gamma \in G(F)$ be regular semi-simple and let $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$ be a normal r-approximation.*

$$\Theta_{\pi_{(S,\theta)}}(\gamma) = \sum_{\substack{g \in J(F)\backslash G(F)/S(F) \\ \gamma^g_{<r} \in S(F)}} \epsilon_{s,r}(\gamma^g_{<r}) \cancel{\epsilon^r(\gamma^g_{<r})} \tilde{e}(\gamma^g_{<r}) \cdot \theta(\gamma^g_{<r}) \widehat{\iota^J_g}_X(\log(\gamma_{\geq r}))$$

$e_{b,0} \cdot e_{\Delta,1} \cdot e_{b,2}$

$$\boxed{\mathrm{ord}_x(\alpha) = \{r \in \mathbb{R} | \mathfrak{g}_\alpha(F_\alpha)_{x,r+} \neq \mathfrak{g}_\alpha(F_\alpha)_{x,r}\},}$$

$$R_\delta = \{\alpha \in R(T,G) \smallsetminus R(T,G^{d-1}) | \alpha(\delta) \neq 1\},$$

$$R_{r/2} = \{\alpha \in R_\delta | r \in 2\mathrm{ord}_x(\alpha)\},$$

$$R_{(r-\mathrm{ord}_\delta)/2} = \{\alpha \in R_\delta | r - \mathrm{ord}(\alpha(\delta)-1) \in 2\mathrm{ord}_x(\alpha)\}.$$

$$t_\alpha = \frac{1}{2} e_\alpha N_{F_\alpha/F_{\pm\alpha}}(w_\alpha)\langle H_\alpha, X\rangle(\alpha(\delta)-1) \in O^\times_{F_\alpha},$$

$$w_\alpha \in F^\times_\alpha \quad , \quad \mathrm{ord}(w_\alpha) = [\mathrm{ord}(\alpha(\delta)-1) - r]/2.$$

$$\mathfrak{G} = q^{-1/2} \sum_{x \in k} \Lambda(x^2) \in \mathbb{C}^\times.$$

$$\epsilon_{\mathrm{s,r}}(\delta) = \prod_{\alpha \in \Gamma\backslash(R_{(r-\mathrm{ord}_\delta)/2})_{\mathrm{s,r}}} \mathrm{sgn}_{F_{\pm\alpha}}(G_{\pm\alpha}) \cdot (-\mathfrak{G})^{f_\alpha} \cdot \mathrm{sgn}_{k^\times_\alpha}(t_\alpha).$$

$$\epsilon^{\mathrm{r}}(\delta) = \prod_{\alpha \in \Gamma \times \{\pm 1\}\backslash(R_{r/2})^{\mathrm{s}}} \mathrm{sgn}_{k^\times_\alpha}(\alpha(\delta)) \cdot \prod_{\alpha \in \Gamma\backslash(R_{r/2})_{\mathrm{s,u}}} \mathrm{sgn}_{k^1_\alpha}(\alpha(\delta)).$$

$$\tilde{e}(\delta) = \prod_{\alpha \in \Gamma\backslash(R_{(r-\mathrm{ord}_\delta)/2})^{\mathrm{s}}} (-1).$$

Main tool: Don't panic!

Try to induct on this formula over

$$G^0 \subset \cdots \subset G^d.$$

PANIC! Error in Yu's construction.

Needed FKS twist.

New formula

Thm (Spice): One can unwind the induction and collect all orbital integrals.

Reinterpret the roots of unity!

Thm: $\mathrm{ord}_x(\alpha) = \begin{cases} e_\alpha^{-1}\mathbb{Z}, & \alpha \text{ ramified} \\ e_\alpha^{-1}\mathbb{Z}, & \alpha \text{ unram}, f(\alpha)=+1 \\ e_\alpha^{-1}(\mathbb{Z}+\frac{1}{2}), & \alpha \text{ unram}, f(\alpha)=-1 \end{cases}$

α is symmetric

$F_\alpha / F_{\pm\alpha}$

$f: R(T,G)_{sym} \to \{\pm 1\}$

Thm: $\Theta_{\pi(S,\theta)}(\gamma) =$

$$e(G)\,e(J)\cdot \varepsilon\left(\tfrac{1}{2}, X^*(T_G)_{\mathbb{C}} - X^*(T_J)_{\mathbb{C}}, \Lambda\right)$$

$$\sum_{\substack{g \in J\backslash G/S \\ \gamma_{<r}^{g} \in S}} \Delta_{II}^{abs}(\gamma_{<r}^{g}) \cdot \theta(\gamma_{<r}^{g}) \cdot \hat{\iota}^{J}_{gX}(\log \gamma_{\geq r})$$

$$\Delta_{II}^{abs} = \prod_{\substack{R(S,G)_{sym}/\Gamma \\ \alpha(\delta) \neq 1}} \chi_\alpha\left(\frac{\alpha(\delta)-1}{\langle X, H_\alpha\rangle}\right)$$

$\uparrow$ θ

Cor: For $\gamma \in S$ top ss,

$$\Theta_{\pi(S,\theta)}(\gamma) = e(G)\cdot \varepsilon\left(\tfrac{1}{2}, X^*(T) - X^*(S), \Lambda\right)\cdot \sum_{w} \Delta_{II}(\gamma^w)\,\theta(\gamma^w).$$

$\chi_\alpha : F_\alpha^\times \to \mathbb{C}^\times$

$\underline{F=\mathbb{R}}$: There is a nat choice of $\pm$

χ s.t.

$$\Theta_{\pi(S,\theta)}^{e^S}(\gamma) = e(G)\,\varepsilon(\tfrac{1}{2}, \chi'(T)\cdot\chi'(S \ldots$$

$$\sum_{w} \Delta_{II}^{abs}(\gamma^w)\cdot\theta(\gamma^w).$$

$$\chi : \mathbb{C}^\times \to \mathbb{C}^\times$$

<u>Covers</u>: Return to $F=\mathbb{R}$

$$\Theta_\pi(\gamma) = (-1)^{q(G)} \sum_{w} \frac{\theta(\gamma^w)}{\prod_{\alpha>0}(1-\alpha(\gamma^w)^{-1})}$$

$$= (-1)^{q(G)} \sum_{w} \frac{(\theta\cdot\rho)(\gamma^w)}{\prod_{\alpha>0}\left(\alpha^{1/2}(\gamma^w) - \alpha^{-1/2}(\gamma^w)\right)}$$

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$$

Numerator, denominator are <u>not</u> functions on S

$$\begin{tikzcd} 1 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d, equal] & S_\rho \arrow[r] \arrow[d, "\rho"] & S \arrow[r] \arrow[d, "2\rho"] & 1 \\ 1 \arrow[r] & \{\pm 1\} \arrow[r] & \mathbb{C}^\times \arrow[r, "()^2"] & \mathbb{C}^\times \arrow[r] & 1 \end{tikzcd}$$

Note: $\theta \cdot \rho$ is regular.

$$\Theta_\pi(\gamma) = e(G) \cdot \varepsilon \cdot \sum_w [a_S \cdot \theta_T](\gamma^w)$$

$$a_S : S_T \to \{\pm 1\}$$

The def of S_T and a_S can be generalized to any local field. Then

$$\Theta_{\pi(S,\theta)}(\gamma) = e(G) \cdot \varepsilon \cdot \sum_w [a_S \cdot \theta_T](\gamma^w)$$

$\forall$ p'-order.