Recall: $f \in \mathcal{C}_c^\infty(G)$, π adm

$$\pi(f)v = \int_G f(x)\,\pi(x)\,v\,dx$$

$$\Theta_\pi(f) = \operatorname{tr} \pi(f)$$

Thm (H.C.): Θ_π is represented by a function ($\Theta_\pi \in L^1_{loc}(G)$)

·) $\Theta_\pi|_{G_{rs}}$ is loc const

·) $\Theta_\pi \cdot |D_G|^{1/2}$ bounded

Note: $\Theta_\pi(f)$ depends on dx, but $\Theta_\pi(g)$ does not.

Notation: In what comes, I will replace Θ_π by $\Theta_\pi \cdot |D_G|^{1/2}$.

<u>Goal today</u>: Regular depth zero supercuspidal rep's

- Construction
- Parameterization
- Characters

Main Ideas: DeBacker – Reeder (2009)

<u>Thm</u> (Moy – Prasad 1996):

π irred d.z. s.c. rep $\iff$

$\pi = \text{c-Ind}_{G_x}^{G} \sigma$, $\quad x \in \mathcal{B}$ vertex

$G_x = \text{Stab}_G(x)$

$\sigma \in \text{Irr}(G_x / G_{x,0+})$

$\sigma|_{G_{x,0}/G_{x,0+}}$ cuspidal

Recall: $\mathcal{B}$ BT Building

$x \in \mathcal{B}$

$$\begin{tikzcd}
G_x^0 \arrow[r, phantom, "\subset"] & G_x^1 \arrow[r, phantom, "\subset"] & G_x \\
G^0 \arrow[u, phantom, "\cap" sloped] \arrow[r, hook] & G^1 \arrow[u, phantom, "\cap" sloped] \arrow[r, hook] & G \arrow[u, phantom, "\cap" sloped]
\end{tikzcd}$$

$$\begin{tikzcd}
G^1 \arrow[dr] & & \\
G^0 \arrow[u, hook] \arrow[r] & G \arrow[r] \arrow[dr] & \pi_1(G)_I^{Fr} \arrow[d] \\
 & & \pi_1(G)^\Gamma
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{G}_x^0 \arrow[r, hook] \arrow[d] & \mathcal{G}_x^1 \arrow[r, hook] \arrow[d] & \mathcal{G}_x \arrow[d] & /O_F \\
\overline{\mathcal{G}}_x^0 \arrow[r, hook] \arrow[d] & \overline{\mathcal{G}}_x^1 \arrow[r, hook] \arrow[d] & \overline{\mathcal{G}}_x \arrow[d] & /k_F \\
\overline{G}_x^0 \arrow[r, hook] & \overline{G}_x^1 \arrow[r, hook] & \overline{G}_x & /k_F \text{ red}
\end{tikzcd}$$

$$[\overline{Z} \cdot \overline{G}_x^0 : \overline{G}_x] < \infty$$

Fact: Let $S \subset G$ max torus, $S' \subset S$ max unramified subtorus. TFAE

1. $S' \subset G$ is a max unram subtorus
2. $S = Z_G(S')$
3. $S \times F^u$ is a min Levi in $G \times F^u$.

Def: Such a torus S is called maximally unramified.

The point of S: Let S be a maximally unramified elliptic max torus. Then we have

$$\mathcal{A}(S' \times F^u) \subset \mathcal{B}(G \times F^u)$$

and $\mathcal{A}(S' \times F^u)^{Fr} = \{x\} \subseteq \mathcal{B}(G)$.

Prop: x is a vertex.

~~Depth~~

$$S \hookrightarrow G$$

$$\mathfrak{S} \hookrightarrow \mathfrak{G}_x \quad /\mathcal{O}_F$$

$$\begin{array}{ccc} \bar{S} & \hookrightarrow & \bar{G}_x \\ \cup & & \cup \\ \bar{S}' & \hookrightarrow & \bar{G}_x^0 \end{array} \quad \text{max torus}$$

$$(\bar{S}^0) = \bar{S}'.$$

$$(\bar{S})_{\bar{k}} = (\bar{Z} \cdot \bar{S}')_{\bar{k}}$$

<u>Depth zero characters</u>:

$\theta : S \to \mathbb{C}^\times$ depth-zero

$\iff \theta|_{S_{0+}} = 1.$

$\Rightarrow \bar{\theta} : \bar{S} \to \mathbb{C}^\times$

<u>Def</u>: $\bar{\theta}$ is regular. if $\theta \neq \theta^{\omega}$,

$\omega \in N_{\bar{G}_x}(\bar{S})/\bar{S}$.

DL induction for disc groups

Choose $\bar{k}$ - Borel $\bar{S}^\circ \subset \bar{B} \subset \bar{G}_x^\circ$

$$Y = \{ g \in \bar{G}_x \mid g^{-1} Fr(g) \in U \cdot Fr(U) \}$$

$\bar{G}_x \curvearrowright H^i_c(Y, \bar{\mathbb{Q}}_\ell) \curvearrowleft \bar{S}$

Thm: If $\bar{\theta}$ is regular, then $H^i_c(\,)_{\bar{\theta}}$ vanishes away from middle degree, where it is an irred cuspidal rep of $\bar{G}_x$.

Def: $\pi = \text{c-Ind}_{G_x}^{G} \sigma$, σ = inflation to G_x of κ $= H^i_c(Y, \bar{\mathbb{Q}}_\ell)_{\bar{\theta}}$ is a reg d.z. supercuspidal rep.

<u>Thm</u>: There is a bij

$$\{\pi \text{ d.s.s.c. rep}\} \leftrightarrow \{(S,\theta)\}\Big/ G\text{-con}$$

where $S \subset G$ ell max unr max torus

$\theta : S \to \mathbb{C}^\times$ reg d.z. char.

<u>Characters</u>: From now on: G splits over a tame ext, char $F = 0$, $p \gg 0$.

<u>Prop</u>: The char of σ at a reg ss. element $\gamma \in G_x$ is

$$(-1)^{r_G - r_S} \cdot \big|\underbrace{Z_{\bar G_x}(\gamma_s)^\circ}_{C}\big| \sum_{\substack{h \in \bar G_x^\circ \\ h^{-1}\gamma_s h \in \bar S}} \bar\theta(h^{-1}\gamma_s h) \cdot \Big| Q^{C}_{h \bar S h^{-1}}($$

where $\gamma = \gamma_s \cdot \gamma_u$ is the top. Jordan decomp

γ_s: finite ~~prop~~ order, prime-to-p

γ_u: pro-p order

<u>Springer conj, Kazhdan's thm:</u>

Q is the Fourier transform of an orbital integral, up to a sign.

<u>Thm</u> (DeBacker-Reeder):

$$\Theta_{\pi(S,\theta)}(\gamma) = (-1)^{r(G)-r(J)} \sum_{\substack{g \in S\backslash G/J \\ {}^g\gamma_s \in S}} \theta({}^g\gamma_s)\, \hat{\iota}^{J}_{X^g}(\log \gamma_u)$$

$J = \mathrm{Cent}_G(\gamma_s)^\circ$

$X \in \mathrm{Lie}^*(S)_0$ reg mod p

<u>Cor</u> : If $\gamma \in G$ is top ss

$$\Theta_{\pi(S,\theta)}(\gamma) = 0, \text{ if } \gamma \notin {}^{G}S$$

$$= (-1)^{r(G)-r(S)} \sum_{w \in N_G(S)/S} \theta(\gamma^w),$$

Fourier transforms :

V fin-dim F-vs, V^* dual

$\Lambda : F \to \mathbb{C}^\times$ non-triv

$\mathcal{C}_c^\infty(V) \to \mathcal{C}_c^\infty(V^*)$

$$\hat{f}(\xi) = \int_V f(x) \Lambda(\langle x, \xi \rangle) dx$$

& $d \in \mathcal{D}(V^*)$, $\hat{d}(f) = d(\hat{f})$.

Orbital integrals:

$\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}^*$ dual

$$x \in \mathfrak{g}^*, \quad O_x(f^*) = \int_{\mathrm{Ad}^*(G)\cdot x} f(y)\,dy.$$

Thm (H.C.):

Let $x \in \mathfrak{g}^*$ be reg ss. The dist

$\hat{O}_x$ is represented by a function $\hat{\mu}_x : \mathfrak{g} \to \mathbb{C}$ which is

- loc const on $\mathfrak{g}_{rs}$
- Bounded after mult by $|D_{\mathfrak{g}}|^{1/2}$.

<u>Def</u>: $\eta_x(\tau) = |Dg(x)|^{1/2} \cdot$
$|Dg(y)|^{1/2} \cdot$
$\hat{\mu}_x(\tau)$.
