Characters of admissible reps

G group, (π, V) rep on fin-dim $\mathbb{C}$-vs, $\Theta_\pi : G \to \mathbb{C}$
$$g \mapsto \operatorname{tr} \pi(g).$$

G p-adic group
(π, V) irreducible rep of G
(smooth)
$\Theta_\pi(g)$ "=" $\operatorname{tr}(\pi(g))$. ↯

<u>Harish-Chandra</u>:
Given admissible (π, V),
$$\Theta_\pi : C_c^\infty(G) \longrightarrow \mathbb{C}$$
$$f \mapsto \operatorname{tr} \pi(f).$$

$$\pi(f) : V \to V$$
$$v \mapsto \int_G f(g)\pi(g)v\,dg$$

$\mathcal{C}_c^\infty(G) = \{ f : G \to \mathbb{C} \mid$
· supp(f) compact
· $\exists K \subset G$ c.o.s. s.t.
f is K-Biinvariant$\}$

$\pi(f) : V \to V^K$ ← fin. dim.

⇒ $\mathrm{tr}(\pi(f))$ is well-defined.

we had to choose a Haar measure
dg.

G is unimodular ⇒ left-inv Haar
$\qquad\qquad\qquad\qquad\qquad$ = right-inv Haar

Distributions:

Def: A distribution on G is a linear functional

$$C_c^\infty(G) \to \mathbb{C}.$$

$D(G) :=$ the space of all dist.

Observation:

$$L^1_{loc}(G) \hookrightarrow D(G)$$

$$\phi \mapsto C_f \mapsto \int_G \phi(g) f(g)\,dg$$

Def: A dist is representable by a function, if it is in the image of $L^1_{loc}(G)$.

Thm (HC): The dist Θ_π is representable by $f \in L^1_{loc}(G)$ which is

- locally constant on G_{rs}
- $\Theta_\pi \cdot |D_G|_F^{1/2}$ bounded function on G

Notation: $g \in G_{rs} \Leftrightarrow Z_G(g)^0$ is a torus
$\Leftrightarrow g$ lies in a unique max torus

$$D_G(g) = \prod_{\alpha \in R(T,G)} (1 - \alpha(g)).$$

<u>Fact</u>: 1) Let $\pi_1, \dots, \pi_n$ distinct irred reps of G. Then $\Theta_{\pi_1}, \dots, \Theta_{\pi_n}$ are lin ind.

2) Two irreps are iso iff have the same character.

3) The character is additive:

$$0 \to \pi_1 \to \pi_2 \to \dots \to \pi_n \to$$

$$\Rightarrow \quad \sum (-1)^i \Theta_{\pi_i} = 0.$$

Parabolic induction:

$P = MN \subset G$ parabolic

$\pi = n\text{-}\mathrm{Ind}_P^G(\sigma)$.

Prop: $\Theta_\pi(f) = \Theta_\sigma(f^{(P)})$

$f^{(P)} \in C_c^\infty(M)$

$$f^{(P)}(m) = \delta_P(m)^{1/2} \int_N \int_K f(k^{-1} m n k)\, dk\, dn$$

where $K \subset G$ is any c.o.s. $(G = PK)$

and

$$\int_G f(g)\, dg = \int_M \int_N \int_K f(mnk)\, dm\, dn\, dk$$

$$\Longrightarrow \quad \Theta_\pi(x) = \sum_{\substack{g \in M\backslash G \\ gxg^{-1} \in M}} \frac{\Theta_\sigma(gxg^{-1})}{|D_{G/M}(gxg^{-1})|^{1/2}}$$

The Steinberg character

G any p-adic group

$P_0 = M_0 N_0 \subset G$ minimal parabolic

$$\text{n-Ind}_{P_0}^{G} \delta_{P_0}^{-1/2} = \text{Ind}_{P_0}^{G} \mathbb{1}_{P_0} = C^\infty(P_0 \backslash G)$$

For any $P_0 \subset P \subset G$ parabolic

$$C^\infty(P \backslash G) \subset C^\infty(P_0 \backslash G)$$

$$\Sigma_0 = \sum_{P_0 \subsetneq P \subset G} C^\infty(P \backslash G) \subset C^\infty(P_0 \backslash G)$$

Def: $St = C^\infty(P_0 \backslash G) / \Sigma_0$.

Fact: St is square-int rep.

The ~~Borel Se~~ Borel - Serre resolution

$$0 \to I_r \to I_{r-1} \to \cdots \to I_0 \to St \to$$

$$I_t = \bigoplus_{P:\,\mathrm{rk}(P)=t} \mathcal{C}^\infty(P\backslash G)$$

$$I_0 = \mathcal{C}^\infty(P_0\backslash G)$$

$$I_r = 1 = \mathcal{C}^\infty(G\backslash G)$$

Ex: $G = SL_2$:

$$\begin{array}{ccccccccc} 0 & \to & I_1 & \to & I_0 & \to & St & \to & 0 \\ & & \| & & \| & & & & \\ & & 1 & & \mathcal{C}^\infty(B\backslash G) & & & & \end{array}$$

$$\Rightarrow \quad \Theta_{St}(g) = (-1)^{\dim(A_0)} \sum_{(M|P)\ \mathrm{std}} (-1)^{\dim(A_M)} \, \delta_P(g)^{-1/2} \, |D_{G/M}(g)|^{-1/2}$$

A_M = split center of M

$$\Theta_\pi(\gamma) = \begin{cases} \frac{1}{2}\operatorname{sgn}_\epsilon(\operatorname{Im}_\epsilon(\gamma))\,\frac{\psi(\gamma)+\psi(\gamma^{-1})}{|D_G(\gamma)|^{1/2}}\left[(-1)^{r+1}+H(\Lambda',k_\epsilon)\right] & \gamma\in T^\epsilon\setminus Z(G)T^\epsilon_{r+} \\ c_0(\pi)+H(\Lambda',k_\epsilon)\frac{\operatorname{sgn}_\epsilon(\eta^{-1}\operatorname{Im}_\epsilon(\gamma))}{|D_G(\gamma)|^{1/2}} & \gamma\in T^{\epsilon,\eta}_{r+} \\ c_0(\pi)+\frac{1}{|D_G(\gamma)|^{1/2}} & \gamma\in A_{r+} \\ c_0(\pi) & \text{otherwise, if } \gamma\in G_{r+} \\ 0 & \text{otherwise, if } \gamma\notin G_{r+}. \end{cases}$$

$$\Theta_\pi(\gamma) = \begin{cases} \dfrac{\mathrm{sgn}_\varpi(\mathrm{Im}_\varpi(\gamma))H(\Lambda', k_\varpi)}{|D_G(\gamma)|^{1/2}}\left\{\psi(\gamma) \ + \ \psi(\gamma^{-1})[\dfrac{\mathrm{sgn}_\varpi(-1)+1}{2}]\right\} & \\ & \gamma \in T^\theta \setminus Z(G)T^\theta_r \\ \dfrac{q^{-1/2}}{2\,|D_G(\gamma)|^{1/2}} \displaystyle\sum_{\substack{\gamma' \in (C_\varpi)_{r:r+} \\ \gamma' \neq \gamma^{\pm 1}}} \mathrm{sgn}_\varpi(\mathrm{tr}_\varpi(\gamma - \gamma'))\psi(\gamma') & \\ \quad + \ \dfrac{1}{2} H(\Lambda', k_\varpi)\,\mathrm{sgn}_\varpi(\eta^{-1}\,\mathrm{Im}_\varpi(\gamma))\dfrac{\psi(\gamma)+\psi(\gamma^{-1})}{|D_G(\gamma)|^{1/2}} & \\ & \gamma \in T^{\varpi,\eta}_r \setminus T^{\varpi,\eta}_{r+} \\ \dfrac{q^{-1/2}}{2\,|D_G(\gamma)|^{1/2}} \displaystyle\sum_{\gamma' \in (C_\varpi)_{r:r+}} \mathrm{sgn}_\varpi(\mathrm{tr}_{\epsilon\varpi}(\gamma) - \mathrm{tr}_\varpi(\gamma'))\psi(\gamma') & \gamma \in T^{\epsilon\varpi,\eta}_r \setminus T^{\epsilon\varpi,\eta}_{r+} \\ c_0(\pi) + H(\Lambda', k_\varpi)\dfrac{\mathrm{sgn}_\varpi(\eta^{-1}\,\mathrm{Im}_\varpi(\gamma))}{|D_G(\gamma)|^{1/2}} & \gamma \in T^{\varpi,\eta}_{r+} \\ c_0(\pi) + \dfrac{1}{|D_G(\gamma)|^{1/2}} & \gamma \in A_{r+} \\ c_0(\pi) & \text{otherwise, if } \gamma \in G_{r+} \\ 0 & \text{otherwise, if } \gamma \notin G_{r+}. \end{cases}$$

Don't panic!

Guiding light of real groups

<u>Thm</u> (H.C.) G real reductive group

1) G has d.s. reps $\Leftrightarrow$
   G has an elliptic max torus

2) $\{\text{d.s. irreps}\} \overset{1:1}{\longleftrightarrow} (S, B, \theta)/_{G\text{-co}}$

   $S \subset G$ ell max torus
   $\theta : S \to \mathbb{C}^\times$ reg " char
   $B \subset G_{\mathbb{C}}$ Borel, $S \subset B$
   $d\theta$ is B-dom.

12

3) The rep π corr to (S, B, θ) is uniquely char by

$$\Theta_\pi(s) = (-1)^{q(G)} \sum_{w \in N_G(S)/S} \frac{\theta(s^w)}{\prod_{\alpha >_B 0} (1-\alpha(s^w)^{-1})}$$

$s \in S$

$\theta : S \to \mathbb{C}^\times$

$\uparrow$

$(S^1)^n \cong S_{sc}$

$\theta \in X^*(S_{sc})$ cut up into Weyl chambers