AWS 2025: Mod-p representations of p-adic groups

Brief description of course and projects

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1. Introduction

Let p be a prime and let $F/\mathbb{Q}_p$ denote a finite extension with uniformizer ϖ. By a *mod-p representation* we will mean a representation of some group over a (fixed) algebraically closed field of characteristic p.

Traditionally, if G is some p-adic reductive group, such as $G = \mathrm{GL}_n(F)$, one studies its representation theory over a field of characteristic zero, such as $\mathbb{C}$, and this is an important part of the classical Langlands program. Just as for finite groups, it is also natural to study modular representations of G. It turns out that the mod-ℓ representation theory of a p-adic group is not *that* different from the complex representation theory if $\ell \neq p$. However, if $\ell = p$ there are fundamental differences.

To give some basic examples: if K denotes an open pro-p subgroup of G (such as $1 + \varpi \mathrm{M}_n(\mathcal{O}_F)$ in $G = \mathrm{GL}_n(F)$), then any nonzero smooth mod-p representation V of G has *nonzero* K-invariants V^K. Also, if V^K is finite-dimensional for such a K, then $V^{K'}$ is finite-dimensional for *any* open subgroup K' of G. Both of these properties fail completely over $\mathbb{C}$. On the other hand, some of the basic concepts over $\mathbb{C}$ do not adapt to mod-p representations, like many of the analytic constructions that depend on Haar measure.

Smooth mod-p representations of p-adic groups were first studied for the group $\mathrm{GL}_2(F)$ by Barthel–Livné in the mid 1990's. Around 2000, Breuil started to investigate the possibility of and find evidence for a mod-p (resp. p-adic) Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. More precisely, a mod-$p$ (resp. p-adic) Langlands correspondence should relate smooth mod-p (resp. certain p-adic) representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ with Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ (resp. Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$), and the two kinds of correspondences should be related to each other by reduction modulo p. Over the following ten or so years, work of Breuil, Colmez, Kisin, Emerton, Paškūnas, ... established such correspondences and studied them in detail for the group $\mathrm{GL}_2(\mathbb{Q}_p)$. However, for other groups, say $\mathrm{GL}_2(F)$ for $F \neq \mathbb{Q}_p$ or $\mathrm{GL}_n(F)$ for $n > 2$, mod-p (and p-adic) Langlands correspondences have so far remained out of reach, despite various positive indications. Even from a purely representation-theoretic point of view it turned out, surprisingly and as will be discussed in the course, that the group $\mathrm{GL}_2(\mathbb{Q}_p)$ is unusually well-behaved and even $\mathrm{GL}_2(F)$ for $F \neq \mathbb{Q}_p$ is much more complicated.

Just as in the classical Langlands program one expects there to be a global mod-p (and p-adic) Langlands correspondence, occurring in the cohomology of Shimura varieties, or more generally, arithmetic quotients of symmetric spaces. For GL_2 over $\mathbb{Q}$ this was realized by Emerton, and for more general groups this expectation allows one to define a smooth mod-p representation associated to any (automorphic) *global* mod-p Galois representation. One hopes that this association realizes the (local) mod-p Langlands correspondence. I will discuss this in the last part of my course, and it will play a role in two of the projects.

2. Course outline

The purpose of this course is to give an introduction to the mod-p representation theory of p-adic groups. Here is a tentative outline of my lectures.

Lecture 1. We will discuss basic properties of smooth and admissible representations, induced representations, differences between mod-p and complex representations, motivation from the hypothetical mod-p Langlands correspondence, the case of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Lecture 2. Serre weights and Hecke algebras, mod-p Satake transform, parabolic induction and supersingularity/supercuspidality, classification of irreducible admissible representations in terms of supersingular representations.

Lecture 3. The group $\mathrm{GL}_2(F)$ for $F/\mathbb{Q}_p$ non-trivial unramified: coefficient systems and the construction of Breuil–Paškūnas leading to "too many" supersingular representations. Ordinary parts and extensions between irreducible representations.

Lecture 4. Global methods in mod-p representation theory: Emerton's local-global compatibility for GL_2 over $\mathbb{Q}$, global candidates for the mod-p Langlands correspondence, and a survey of recent progress, especially for $\mathrm{GL}_2(F)$.

3. Background

Knowing about local fields and representations of finite groups over $\mathbb{C}$ is essential. Modular representation theory of finite groups is not essential but can be helpful. References are [Ser79], [Ser77].

If you would like to read ahead, you could look at some of these course notes: [Hera], [Bre], [Herb].

4. Project outlines

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Here are some projects I have in mind.

In all projects I use the following notation: $G := \mathrm{GL}_n(F)$, $K := \mathrm{GL}_n(\mathcal{O}_F)$, Z the center of G, T the diagonal torus of G, and U the upper-triangular unipotent subgroup. All representations will be over a fixed algebraically closed field of characteristic p, e.g. $\overline{\mathbb{F}}_p$. A *Serre weight* is an irreducible smooth representation of K, or equivalently of $\mathrm{GL}_n(k_F)$, where k_F denotes the residue field of F. Throughout, σ denotes a Serre weight. Finally, c-Ind denotes compact induction.

4.1. **Satake transforms.** Recently, using derived techniques, Heyer [Hey23, §4.3] defined mod-p Satake transforms which are algebra homomorphisms

$$\mathcal{S}^i : \mathrm{End}_G(\text{c-Ind}_K^G \sigma) \to \mathrm{End}_T(\text{c-Ind}_{K\cap T}^T H^{\dim U - i}(K \cap U, \sigma))$$

for $0 \le i \le \dim U$, where dim here denotes the dimension as a p-adic Lie group. The two extreme cases were studied in detail earlier, starting with [Her11b], [Her11a, §2, §5]. This project will investigate Heyer's Satake transforms in the remaining cases. Useful background reading: [Her11b, §1–2]. Knowledge of group cohomology is essential. Some basics on derived categories, say the first six sections of [Yek], can be helpful.

4.2. **Supersingular representations.** So far, extremely little is known about irreducible admissible supersingular representations of $\mathrm{GL}_n(F)$ for $n > 2$. In fact, to the best of my knowledge, only their existence is known, cf. [HKV20]! The goal of this project is to show that such representations exist with some prescribed properties. This will most likely require global arguments. Therefore, some background on Galois representations associated to automorphic representations (at the very least for classical modular forms) would be very useful. In this project you will be working in the context of unitary groups (see e.g. [EGH13]).

4.3. **Universal supersingular modules.** The universal supersingular module is the quotient $\mathcal{U}(\sigma) := (\text{c-Ind}_{KZ}^G \sigma)/(T_1, \dots, T_{n-1})$, where $T_i \in \text{End}_G(\text{c-Ind}_{KZ}^G \sigma)$ is the Hecke operator supported on the double coset KZt_iKZ, where we write $t_i := \text{diag}(\varpi, \dots, \varpi, 1, \dots, 1)$ (i copies of ϖ followed by $n - i$ copies of 1). It is a smooth representation of G.

For $n = 2$ it is known that $\mathcal{U}(\sigma)$ is admissible iff $F = \mathbb{Q}_p$ (see e.g. [Bre03], [Sch14], [Hen19]).

The goal of this project is to show that $\mathcal{U}(\sigma)$ is at least of infinite length in some cases when $n = 3$. One possible approach is global, in which case the background is similar to the one of the previous project. Another possible approach may be local, but a priori it is less clear to me how doable this is.

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