Global

(for simplicity: GL_2).

Idea: $E/\mathbb{Q}$ finite

G/E reductive gp.

auto. forms on G:

$$\text{“}\,\mathcal{C}\big(G(E)\backslash G(\mathbb{A}_E)/U^p,\ \mathbb{C}\big)\,\text{”}$$

$$\circlearrowleft$$

$$\cancel{G(\mathbb{A}_E^\infty)}$$

$$G(E_p)$$

"replace $\mathbb{C}$ with $\overline{\mathbb{F}}_p$".

For us:

$E/\mathbb{Q}$ tot. real, p inert

D ... quat. alg. $/E$ s.t.

$D \otimes_E E_v$ nonsplit $\forall v | \infty$.

$(\cong \mathbb{H})$

"$G = D^\times$"

adelic gp:

$$\mathbb{A}_E = \prod_v{}' E_v$$

$$= \mathbb{A}_E^\infty \times \prod_{v|\infty} \mathbb{R}$$

$$(D\otimes_E \mathbb{A}_E^\infty)^\times = \prod_{v\nmid\infty}{}' (D\otimes_E E_v)^\times .$$

Level: $\quad \rotatebox{90}{$\leq$} \qquad\qquad\qquad \rotatebox{90}{$\leq$}$

$$\underset{\substack{\text{cpt.}\\ \text{open}}}{U} = \prod_{v\nmid\infty} U_v .$$

$$= \underbrace{U^p}_{v\neq p} \times U_p .$$

$$S(\underset{\substack{\uparrow\\ \text{field}}}{U, k}) := \left\{ f : \overbrace{D^\times \backslash (D\otimes_E \mathbb{A}_E^\infty)^\times / U}^{\text{finite}} \longrightarrow k \right\}.$$

$$S(U^p, k) := \varinjlim_{U_p} S(U^p U_p, k)$$

$$= \left\{ D^\times \backslash (D \otimes_E \mathbb{A}_E^\infty)^\times / U^p \longrightarrow k \ \text{loc. const} \right\}$$

$$\circlearrowleft$$

$(D \otimes_E E_p)^\times$ right transl.

<u>adm.</u> + <u>smooth.</u>:

$$S(U^p, k)^{U_p} = S(U^p U_p, k) \ \text{f.d.}$$

<u>Assume D splits at p</u>

$$\Rightarrow (D \otimes_E E_p)^\times = GL_2(E_p) =: G$$

Hecke actions:

$$\underset{\text{finite}}{\Sigma} := \{ v \mid v \mid p\infty \text{ or } (D \otimes_E E_v)^\times \not\cong GL_2(E_v) \text{ or } U_v \neq GL_2(\mathcal{O}_v) \}$$

For $v \notin \Sigma$:

$$\mathcal{H}_v := k\left[GL_2(\mathcal{O}_v) \backslash GL_2(E_v) / GL_2(\mathcal{O}_v) \right]$$

$$= k[T_v, S_v^{\pm 1}]$$

$$\underset{T_v}{\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix}}, \quad \underset{S_v.}{\begin{pmatrix} \varpi_v & \\ & \varpi_v \end{pmatrix}}.$$

$$\mathbb{T} := \bigotimes_{v \notin \Sigma}{}' \mathcal{H}_v = k[T_v, S_v^{\pm 1} : v \notin \Sigma].$$

$$\begin{array}{ccc} & S(U^p, k) & \\ \circlearrowright & & \circlearrowleft \\ G & & \mathbb{T} \end{array}$$

<u>commutes.</u>

Say $f \in S(U^p, k)$ $\mathbb{T}$-eigenvec.:

$$\begin{cases} T_v f = \lambda_v f \\ S_v f = \mu_v f. \end{cases}$$

Fact: If $k = \overline{\mathbb{Q}_p}$ / or $\overline{\mathbb{F}_p}$,

get unique semisimple cts.

Galois rep.

$\rho_f : \mathrm{Gal}(\overline{E}/E) \longrightarrow GL_2(k)$.

s.t.

(i) ρ_f unram. at all $v \notin \Sigma$.

(ii) $\forall v \notin \Sigma$, $\rho_f(\mathrm{Frob}_v)$

has char. poly. $X^2 - \lambda_v X + q_v \mu$

[(iii) $\rho_f(c) \sim \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. "odd"]

$\uparrow$ cx. conj.

Conversely: given s.s.

$$\bar{r} : \mathrm{Gal}(\bar{E}/E) \longrightarrow GL_2(\overline{\mathbb{F}}_p)$$

s.t. (i) – (iii) hold for some

$\lambda_v, \mu_v, \in k = \overline{\mathbb{F}}_p$, then can

define max. ideal

$$\mathfrak{m}_{\bar{r}} = (T_v - \lambda_v, S_v - \mu_v) \triangleleft \mathbb{T}$$

$$\rightsquigarrow \underline{\underline{\pi(\bar{r})}} := S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\bar{r}}]$$

$$\subset S(U^p, \overline{\mathbb{F}}_p).$$

$\circlearrowleft$

G. (adm. sm.)

Hope (rough):

$$\pi(\bar{r}) \overset{??}{\longleftrightarrow} \bar{r}|_{D_p}$$

D_p decomp. gp.

under mod p LLC.

$E = \mathbb{Q}$: ~ proved by Emerton (2011)

First evidence:

Assume $\bar{r}$ modular,

i.e. $\pi(\bar{r}) \neq 0$.

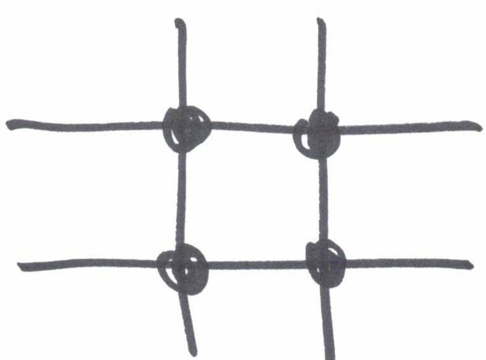
Weight part of Serre's conj.

(BDJ, GLS, GK, ...)

$$\mathrm{soc}_K \, \pi(\bar{r}) = \bigoplus_{\sigma \in W(\bar{r})} \sigma .$$

where $W(\bar{r})$ is a finite set that only depends on $\bar{r}|_{D_p}$.

Picture: $f = 2$, $\bar{r}|_{D_p}$ semisimple



hypercube. (in ext. gph)

K_1-invariants

(EGS, LMS, HW, L)

$\pi(\bar{F})^{K_1} \cong D_0(\bar{F})$ as Γ-rep.

$K/K_1 = \Gamma$

$:=$ largest Γ-rep. s.t.

$\mathrm{soc}_K D_0(\bar{F}) = \bigoplus_{\sigma \in W(\bar{F})} \sigma$ <u>and</u>

$[D_0(\bar{F}) : \sigma] = 1$.

<u>Rk:</u> BP defined $D_0(\bar{F})$ $\circlearrowleft W$
& diag. $(D_0(\bar{F}), D_0(\bar{F})^{I_1}, \mathrm{incl})$

Picture.

$D_0(\bar F)$.

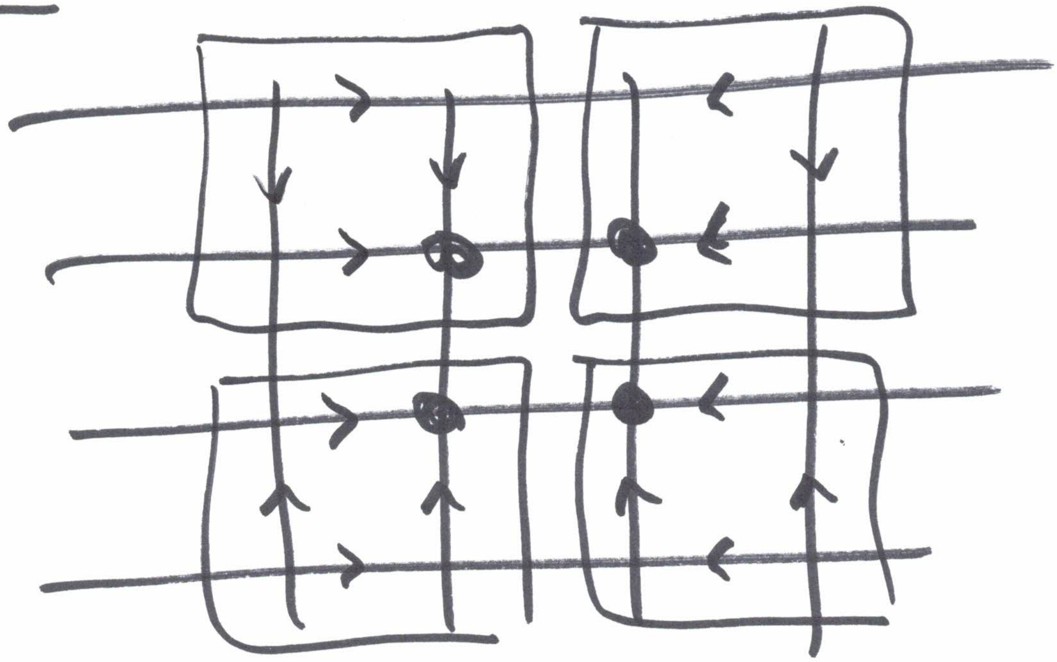


Diagram (DL)

The diagram $\overbrace{(\pi(\bar F)^{K_1}, \pi(\bar F)^{I_1}, \text{incl.}}^{\text{of BP type.}}$

only depends on $\bar F|_{D_p}$.

Main input: Taylor-Wiles method

Existence of (adm.) supersing. reps. (HKV).

Idea: show $\exists \Gamma$ s.t.

$\pi(\Gamma)$ contains a ss'c subrep.

$\mathrm{Hom}_K(V, \pi(\bar{r}))$

$\circlearrowleft$

$\varphi_{1,0} \in \mathcal{H}_G(V)$.

nilpotent

(eval = 0)

$\mathrm{Hom}_K(V, S(U^p, \bar{\mathbb{F}}_p)[m_{\bar{r}}]$

$\|$

$S(U, \underbrace{V^\vee}_{\text{dual}})[m_{\bar{r}}]$

$(U_p = GL_2(\mathcal{O}_p))$

lift π-evals.

$\tilde{V}$ = lift of V.

$S(U, \tilde{V}^\vee)$

classical mod. forms

$S(U, \tilde{V}^\vee)^{[p_r]}$. evals.

$r : \mathrm{Gal}(\overline{E}/E) \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$.
lifting $\overline{r}$.

$r|_{D_p}$ crystalline
+ Hodge-Tate wts.
$\leftrightarrow \tilde{V}$.

① construct. modular $\overline{r}$ s.t
$\overline{r}|_{D_p}$ irred.

② If $\varphi_{1,0}$ has a nonzero eval.
$\Longrightarrow r|_{D_p}$ is reducible

LLLM = Le-le Hung-Levin-Morra

BDJ = Buzzard-Diamond-Jarvis

GLS = Gee-Liu-Savitt

GK = Gee-Kisin

EGS = Emerton-Gee-Savitt

LMS = Le-Morra-Schraen

HW = Hu-Wang

BP = Breuil-Paškūnas

DL = Dotto-Le

HKV = H.-Koziol-Vignéras