(Too many) supersing. reps. of $GL_2(F)$,

$F/\mathbb{Q}_p$ unram., deg f

$(q = p^f)$, $p > 2$

$G = GL_2(F)$

$K := GL_2(\mathcal{O}_F)$

$\cup$

$I := \{ g \in K : g \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod p$

Iwahori

$\triangledown$

$I_1 := \{ \cdots\cdots\cdots \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \bmod p$

pro-p (Sylow)

$$I / I_1 \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$$

$$Z := Z(G) = \left\{ \begin{pmatrix} x & \\ & x \end{pmatrix} : x \in F^\times \right\}$$

<u>Idea</u> $G \curvearrowright \mathcal{T}$ = Bruhat-Tits tree

vertices:

$$\left(\mathcal{O}_F\text{-lattices } \Lambda \subset F^{\oplus 2} \right) / F^\times$$

edges:

$$[\Lambda] - [\Lambda']$$

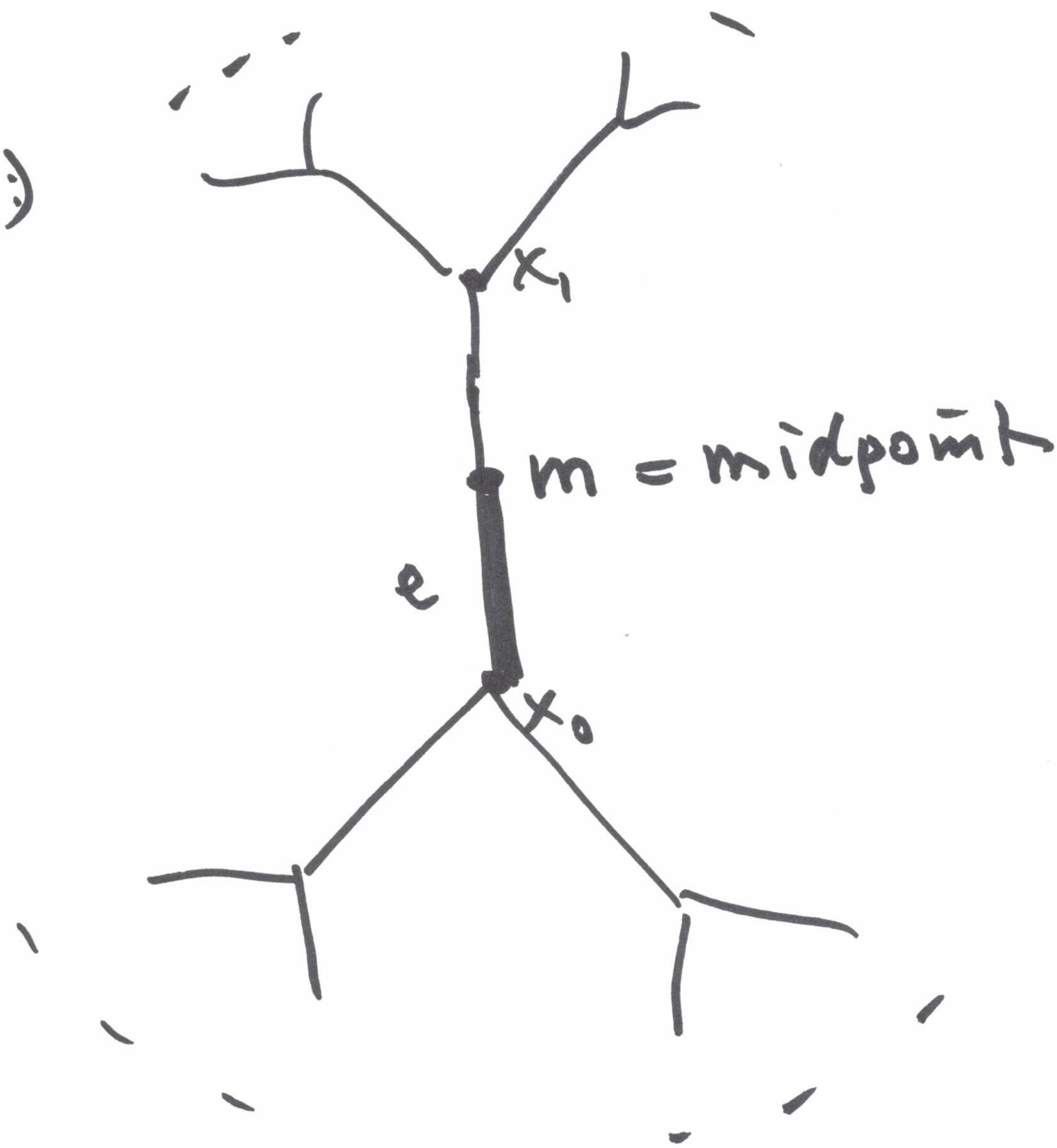
$$\Leftrightarrow \quad p\Lambda \subsetneq \Lambda' \subsetneq \Lambda.$$

action <u>transitive</u>

$$x_0 := [\mathcal{O}_F^{\oplus 2}]$$

$$x_1 := [\mathcal{O}_F \oplus p\,\mathcal{O}_F] = \begin{pmatrix} 1 & \\ & p \end{pmatrix} \cdot x_0$$

$(q=2:)$



$$\mathrm{Stab}_G(x_0) = KZ$$

$$\mathrm{Stab}_G(\overset{e}{\cancel{m}}) = IZ$$

$$\mathrm{Stab}_G(m) = IZ \sqcup IZ\underbrace{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}_{x_0 \leftrightarrow x_1}$$

$$=: \mathcal{N}$$

$$\underset{\text{Serre}}{\rightsquigarrow} \quad \boxed{KZ \ast_{IZ} \mathcal{N} \xrightarrow{\sim} G}$$

$$\begin{array}{c} \| \\ \text{pushout} \end{array}$$

Def: A __diagram__ (D_0, D_1, r)

- D_0 ... smooth KZ-rep.
- D_1 ... smooth $\mathcal{N}$-rep.
- $r : D_1 \hookrightarrow D_0$ inj. IZ-linear
- D_0 adm. + $p \in Z$ acts trivially.

__Rk__: often, $D|_I = \underset{\substack{\cup \\ I}}{D_0^{I_1}}$

$\underline{\text{Ex}}$: If π adm. G-rep. ($p \in Z$ trivial)

$$\begin{array}{ccc} \left(\pi|_{KZ}, \right. & \pi|_{N}, & \left. \text{incl}\right) \\ & \cup & \\ \left(\pi^{K_1}, \right. & \pi^{I_1}, & \left. \text{incl.}\right) \\ \circlearrowleft & \circlearrowleft & \\ KZ & N & \end{array}$$

diagram.

$\underline{\text{Def}}$: $\mathrm{soc}_K \pi$ = largest semisimple K-subrep. of π ($\subset \pi^{K_1}$ f.d.)

(K-socle)

Thm (Paškūnas)

Given (D_0, D_1, r)

$\exists$ adm. G-rep. π s.t.

(i) $(D_0, D_1, r) \subset (\pi|_{KZ}, \pi|_{N}, \text{incl})$ subdiag.

(ii) $\mathrm{soc}_K D_0 = \mathrm{soc}_K \pi$.

(iii) $\pi = \langle G \cdot D_0 \rangle$

Moreover, if (D_0, D_1, r) irred, π is irred.

highly non-can!

<u>Idea:</u>

$$\pi' := \mathrm{inj}_K D_0$$

= "largest sm. K-rep. s.t. K-socle $= \mathrm{soc}_K D_0$".

(unique up to isom.)

Action of I on π' extends to $\mathcal{N}$

$$\begin{array}{c} \pi' \\ \cup \\ D_0 \\ \cup \\ D_1 \circlearrowleft \mathcal{N} \end{array}$$

$\begin{pmatrix} p & \\ & 1 \end{pmatrix} \in \mathcal{N}$

amalgam

$\rightsquigarrow$ π' is a G-rep.

(i), (ii) ✓

$\pi := \langle G \cdot D_0 \rangle \subset \pi'$. //

Applications

Aside on mult. free reps.

H gp.

W f.d. H-rep.

W is **multiplicity free** if all J-H factors occur with mult. $= 1$.

<u>Fact</u> (exercise)

$\exists$ partial order $\leq$ on $JH(W)$

s.t.

$$\left\{\begin{matrix}H\text{-subrep.}\\ \text{of } W\end{matrix}\right\} \xhookrightarrow{1:1} \left\{\begin{matrix}X \subset JH(W): \\ a \leq b,\ b \in X \\ \Rightarrow a \in X\end{matrix}\right\}$$

$$W' \longmapsto JH(W').$$

Hasse diag.

$$JH(W), \leq \;=\; \{a, b, c, d\}$$

$$\begin{tikzcd}
 & & d \arrow[d] \\
b \arrow[dr] & & c \arrow[dl] \\
 & a &
\end{tikzcd}$$

$a \leq b$

$\vdots$

$\Rightarrow \mathrm{soc}_H W \cong a$

$(\mathrm{cosoc}_H W \cong b \oplus d)$.

<u>Extension graph</u> (LLLM).

$\Gamma = GL_2(\mathbb{F}_q)$

vertices = Serre weights.

edges $V - V'$

$\Leftrightarrow \mathrm{Ext}^1_\Gamma(V, V') \neq 0$

$\Leftrightarrow$ - - - - $^{\prime\prime}(V', V)$ - - - -

(subgraph $\mathbb{Z}^f$).

$\underline{f=1}$:

write (r) for $\mathrm{Sym}^r(\mathbb{C}^2)$ $\left(\otimes \det^{\cdots}\right)$

$0 \leq r \leq p-1$

$\cdots - (p-1-r) - (r) - (p-3-r) - (r+2)$

$- \cdots$

$\underline{f=2}$: (r_0, r_1) for

$\mathrm{Sym}^{r_0}(\mathbb{C}^2) \otimes \mathrm{Sym}^{r_1}(\mathbb{C}^2)^{\cdots}$

$\left(\otimes \det^{\cdots}\right)$

$0 \leq r_0, r_1 \leq p-1$

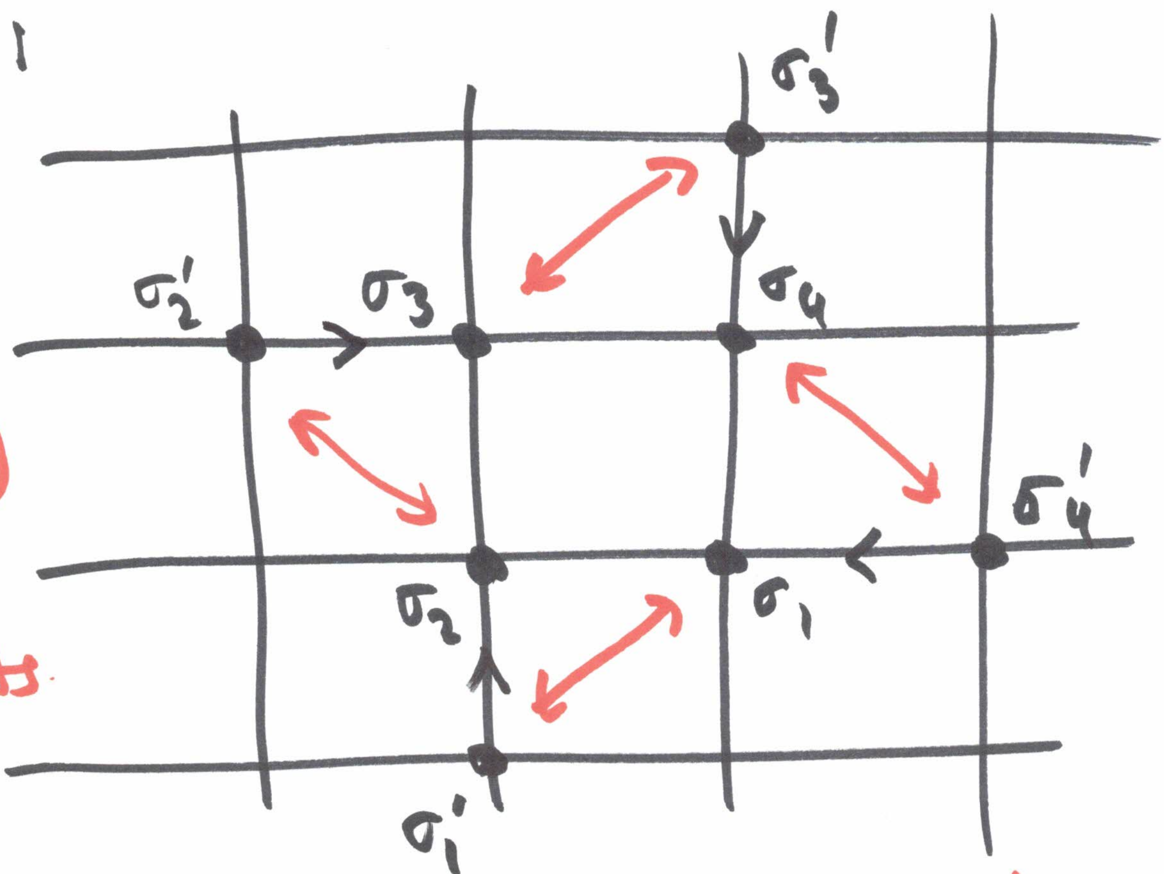
$$
\begin{tikzcd}[arrows=dash]
 & & \arrow[d] & \arrow[d] \\
 & \arrow[r] & (p-2-r_0,\, r_1+1) \arrow[r] \arrow[d] & (p-3-r_0,\, p-3-r_1 \arrow[d] \\
 & (r_0-1,\, p-2-r_1) \arrow[r] \arrow[u] \arrow[d] & (r_0,\, r_1) \arrow[r] \arrow[d] & (r_0+1,\, p-2-r_1) \arrow[d] \\
 & \arrow[r] & (p-2-r_0,\, r_1-1) \arrow[r] \arrow[d] & {} \\
 & & {} &
\end{tikzcd}
$$

Ex: supersing. reps. for $f = 2$

need (D_0, D_1, r)

$D_1 = D_0^{I_1}$, $r = $ incl.

D_0:



$\begin{pmatrix} & 1 \\ p & \end{pmatrix}$ on I_1-invts.

ext. graph!

ie.

$$D_0 = \begin{pmatrix} \sigma_4' \\ \downarrow \\ \sigma_1 \end{pmatrix} \oplus \begin{pmatrix} \sigma_1' \\ \downarrow \\ \sigma_2 \end{pmatrix} \oplus \cdots$$

moreover, $\overset{\text{1-dim}}{\sigma_i^{I_1}}$ $\qquad (\sigma_i')^{I_1}$

$$\sigma_i^{I_1} \circlearrowleft I/I_1 = \mathbb{F}_q^\times \times \mathbb{F}_q^\times$$

$$\sigma_i^{I_1} \xleftrightarrow{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}} (\sigma_i')^{I_1}$$

8-dim

$$D_1 := D_0^{I_1} = \oplus\left(\sigma_i^{I_1} \oplus (\sigma_i')^{I_1}\right).$$

Pick bases v_i $\quad v_i'$.

Define N-action on D_1:

$$\begin{pmatrix} & 1 \\ p & \end{pmatrix} : v_i \longmapsto \begin{cases} v_i' & i \neq 4 \\ \lambda v_i' & i = 4 \end{cases}$$

$\lambda \in \mathbb{C}^\times$.

$D(\lambda) := (D_0, D_1, \text{incl.})$

$\rightsquigarrow$ irred. adm. G-rep. π_λ

Thm.

$\mathrm{soc}_K \pi_\lambda = \sigma_1 \oplus \cdots \oplus \sigma_4$.

$\Rightarrow$ supersing. $\cancel{(\#)}$ by B–L.

$\pi_\lambda \cong \pi_{\lambda'} \Rightarrow D(\lambda) \cong D(\lambda') \Rightarrow \lambda = \lambda'$.