$\boxed{\text{Correction:}}$ Serre weights

$\Gamma = GL_2(\mathbb{F}_q) \curvearrowright \mathrm{Sym}^r(\mathbb{C}^2)$, $0 \le r \le q-1$

<u>not</u> always irred. if $q > p$.

Instead

$$V_r := \langle \Gamma \cdot Y^r \rangle \subset \mathrm{Sym}^r(\mathbb{C}^2)$$

irred.

$$\left(\simeq \bigotimes_{i=0}^{f-1} \left(\mathrm{Sym}^{r_i}(\mathbb{C}^2)\right)^{\varphi^i} \right.$$

$$\left. \text{where } r = \sum_{i=0}^{f-1} r_i p^i \right)$$

Hecke algebras

C arb. field

$H \leq G$ cpt. open

V f.d. rep. of H (over C)

π smooth G-rep.

$$\Rightarrow \operatorname{Hom}_H(V, \pi|_H)$$

$$= \operatorname{Hom}_G(\underbrace{c\text{-}\operatorname{Ind}_H^G V}_{\cong C[G] \otimes_{C[H]} (-)}, \pi)$$

$\circlearrowright$ (right)

$$\mathcal{H}_G(V) = \mathrm{End}_G(c\text{-}\mathrm{Ind}_H^G V)$$

Hecke alg. ring.

Lemma: $\exists$ alg. isom.

$$\mathcal{H}_G(V) \cong \{ \varphi : G \longrightarrow \mathrm{End}_C(V) :$$

- $\varphi(h_1 g h_2) = h_1 \circ \varphi(g) \circ h_2 \quad \forall h_i \in H,\ g \in G.$
- φ cpt. supp. $\}$

(under convo.)

<u>Rk:</u> If $V = \mathbb{1}$,

$$\mathrm{Hom}_H(\mathbb{1}, \pi) = \pi^H$$

$$\circlearrowleft$$

$$\mathcal{H}_G(\mathbb{1})$$

$$\shortparallel$$

$$\left\{ \underset{\text{finite supp.}}{\varphi : H\backslash G/H \longrightarrow \mathbb{C}} \right\}.$$

(double coset alg.)

<u>Idea</u>(Lemma):

$$\overset{\text{Hom}}{\cancel{\text{End}}}_G(c\text{-Ind}_H^G V, c\text{-Ind}_H^G V)$$

$$\underset{\text{Frob.}}{=} \quad \text{Hom}_H(V, c\text{-Ind}_H^G V)$$

$$\text{"}V \longrightarrow (G \longrightarrow V)\text{"}$$

$$\longleftrightarrow \text{"}G \longrightarrow (V \longrightarrow V)\text{"} \longleftrightarrow \text{RHS.}$$

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If $C = \mathbb{C}$, $H = K$ (max. cpt.)

$$V = \mathbb{1}$$

there's Satake isom.

$$\mathcal{H}_G^{\mathbb{C}}(\mathbb{1}) \xrightarrow{\sim} \mathcal{H}_T^{\mathbb{C}}(\mathbb{1})^{W = S_n}$$

$$\varphi \longmapsto \Big(t \longmapsto \underbrace{\delta(t)^{-1/2}}_{\substack{\text{modulus} \\ \in p^{\mathbb{Z}}}} \cdot \sum_{(U \cap K) \backslash U} \varphi(ut) \Big)$$

Back to C of char. p.

$$H = K = GL_n(\mathcal{O}_F).$$

Mod p Satake

$$V \ldots \text{Serre weight} \curvearrowleft \Gamma$$

Fact $V_{U\cap K}$ (co-invts). $\cong V^{\bar{U}\cap K}$

$$\circlearrowleft$$

$$T\cap K$$

$\bar{U} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$.

is 1-dim. (an irrep. of $T\cap K$).

Write $p_U : V \twoheadrightarrow V_{U\cap K}$

Satake:

$$\int^G : \mathcal{H}_G(V) \longrightarrow \mathcal{H}_T(V_{U\cap K})$$

$$\varphi \longmapsto \Big(t \longmapsto \sum_{(U\cap K)\backslash U} \underbrace{p_U \circ \varphi(ut)}\Big)$$

$$\begin{array}{ccc} V & \longrightarrow & V_{U\cap K} \\ {\scriptstyle p_U}\downarrow & \nearrow & \\ V_{U\cap K} & & \end{array}$$

Let $T^+ := \left\{ \begin{pmatrix} x & \\ & y \end{pmatrix} : \underset{\mathrm{val}(x) \geq \mathrm{val}(y)}{*} \right\}$

Thm. (H.)

S^G is an inj. alg. homo.

with image $\mathcal{H}_T^+ := \left\{ \psi \in \mathcal{H}_T : \mathrm{supp}\, \psi \subset T^+ \right\}$

Idea $(n=2)$:

⓪ S^G is alg. homo.

① Cartan: $G = \coprod_{\substack{a \geq b \\ (\text{in } \mathbb{Z})}} K \begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix} K$

Show:

$\forall a \geq b \ \exists! \ \varphi_{a,b} \in \mathcal{H}_G$ s.t.

- $\operatorname{supp} \varphi_{a,b} = K\begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix}K$
- $\varphi_{a,b}\begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix} \in \operatorname{End}_C(V)$ idempotent (= proj.)

$$\Rightarrow \mathcal{H}_G = \bigoplus_{a \geq b} C \cdot \varphi_{a,b} \qquad C\text{-basis.}$$

Easier: $T = \coprod_{a,b \in \mathbb{Z}} (T \cap K)\begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix}$

$$\underset{\cup}{\mathcal{H}_T} = \bigoplus_{a,b} C \cdot \psi_{a,b}$$

$$\mathcal{H}_T^+ = \bigoplus_{a \geq b} C \cdot \psi_{a,b}.$$

② Show $S^G(\mathcal{H}_G) \subset \mathcal{H}_T^+$

③ Triangular arg.

$$S^G(\varphi_{a,b}) = \varphi_{a,b} + \sum_{\substack{a > a' (\geq b') \\ a+b = a'+b' \\ (\text{fin.})}} C \cdot \varphi_{a',b'}$$

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[Cor: $\mathcal{H}_G(V)$ is comm.

If π any adm G-rep,
V Serre weight

$$\underbrace{\begin{array}{c} \mathrm{Hom}_K(V,\pi) = \mathrm{Hom}_K(V,\pi^{K_1}) \\ \circlearrowleft \qquad\qquad\qquad\qquad \underline{\text{f.d.}} \\ \mathcal{H}_G(V) \text{ comm.} \end{array}}$$

$= \bigoplus$ (gen. eigenspaces)

Lemma: If $\theta : T \to \mathbb{C}^\times$ (sm.), then actions

$$\begin{array}{ccc} \mathrm{Hom}_K(V, \mathrm{Ind}_B^G \theta) & \underset{\text{Frob.}}{\cong} & \mathrm{Hom}_{T\cap K}(V_{U\cap K}, \theta) \\ \circlearrowleft & & \circlearrowleft \\ \mathcal{H}_G(V) & \xrightarrow[\mathcal{S}^G]{} & \mathcal{H}_T(V_{U\cap K}) \\ \varphi_{1,0} & \longmapsto & \psi_{1,0} \end{array}$$

Cor. 1: $\mathrm{Ind}_B^G \theta \cong \mathrm{Ind}_B^G \theta'$.

$\Rightarrow \theta = \theta'$

(compare V's + Hecke evals.)

Cor. 2: $\underset{\text{"}T\text{"}}{\varphi_{1,0}} \in \mathcal{H}_G(V)$ acts invertibly on $\mathrm{Hom}_K(V, \mathrm{Ind}_B^G \theta)$

(point: $S^G(\varphi_{1,0}) = \underset{\text{invt. in } \mathcal{H}_T}{\varphi_{1,0}}$)

Def.: An irred. adm. rep. π of G is supersingular $_{(n=2)}$ if $\varphi_{1,0}$ acts nilpotently on $\mathrm{Hom}_K(V, \pi)$ $\forall V$ ($\Leftrightarrow$ eval 0).

Suppose π irred. sm. rep. of G.

Pick $V \subset \pi|_K$.

$\Rightarrow \mathrm{Hom}_K(V, \pi)$ contains

an $\mathcal{H}_G(V)$-eigenvec. $\quad f : V \longrightarrow \pi|_K$

evals: $\chi : \mathcal{H}_G(V) \to \mathbb{C}$

$\Rightarrow$ $\text{c-Ind}_K^G V \twoheadrightarrow \pi$ (surj. as π irred)

Frob

$\circlearrowleft$

$\varphi = \chi(\varphi)$

$\Rightarrow \underbrace{\text{c-Ind}_K^G V \otimes_{\mathcal{H}_G(V), \chi} \mathbb{C}}_{} \twoheadrightarrow \pi$

(G-lin.)

If π is s.s., $\chi(\varphi_{1,0}) = 0$.

<u>Thm.</u> (Breuil, 2003)

If $\boxed{G = GL_2(\mathbb{Q}_p)}$, then

$\forall V\ \forall \chi$, $\chi(\varphi_{1,0}) = 0$,

$\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V),\chi} C$ is irred. + adm.

$\rightsquigarrow$ classify irred. ss. repr. of $GL_2(\mathbb{Q}_p)$.

Thm. (H.)

If $P = P_{n_1,\dots,n_r}$ and

σ_i irred. adm. rep. of $GL_{n_i}(F)$

$\forall i$:

(a) σ_i ss. and $n_i > 1$, **or**

(b) $\sigma_i = \overbrace{Sp_{Q_i}}^{\text{gen. Steinberg}} \otimes (\eta_i \circ \det)$

$\begin{pmatrix} Q_i \subset GL_{n_i}(F) \\ \eta_i : F^\times \to \mathbb{C}^\times \text{ sm.} \end{pmatrix}$

and $\eta_i \neq \eta_{i+1}$ (if σ_i, σ_{i+1} of type (b))

$\Rightarrow \mathrm{Ind}_P^G(\sigma_1 \otimes \dots \otimes \sigma_r)$ irred. adm.