Mod p reps. of p-adic groups

$F / \mathbb{Q}_p$ finite

$\mathcal{O}_F / (\varpi) \cong \mathbb{F}_q \quad (q = p^f)$

$G := GL_n(F)$

$K := GL_n(\mathcal{O}_F)$ max. cpt.

$\triangledown$

$K_1 := 1 + \varpi M_n(\mathcal{O}_F)$

$\triangledown$

$K_2 := 1 + \varpi^2 M_n(\mathcal{O}_F)$

$\triangledown$

$\vdots$

(define top.) cpt. open

pro-p

$$K_r / K_{r+1} \cong M_n(\mathbb{F}_q)$$

$$1 + \varpi^r A \longmapsto \overline{A}$$

$C = \overline{C}$ coeff. field, char. p

smooth / adm.: as in other lectures.

$$V = \bigcup_{\substack{H \leq G \\ \text{(cpt.) open}}} V^H \qquad \dim_C V^H < \infty \quad \forall H.$$

Motivation: Local Langlands

classical

irred. sm. reps. of G over $\mathbb{C}$

$\updownarrow$

"Galois reps.

$\mathrm{Gal}(\overline{F}/F) \longrightarrow GL_n(\mathbb{C})$"

mod p

sm. reps. of G over $\overline{\mathbb{F}}_p$

$\updownarrow$??

Galois reps.

$\mathrm{Gal}(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{F}}_p)$

Challenges

G has no $\mathbb{C}$-valued Haar measure!

$$\left(\int_{K_r} = p^{\cdot\cdot} \int_{K_{r+1}} = 0 \right)$$

$\rightsquigarrow$ no analytic tools!

⚠ If $H \leq G$ open cpt.,

H does not act semisimply!

+ $V \mapsto V^H$ is not exact.

Warmup ($n=1$)

$$G = GL_1(F) = F^\times$$
$$= \varpi^{\mathbb{Z}} \times \mathbb{F}_q^\times \times \underbrace{(1+\varpi\mathcal{O}_F)}_{K_1}.$$

abelian

$\Rightarrow$ an irr. sm. G-rep is 1-dim.

$$\chi : G \longrightarrow C^\times$$

smooth $\Leftrightarrow$ $\ker\chi$ is open

$\Rightarrow$ $\chi(1+\varpi\mathcal{O}_F)$ finite p-gp.

$(*)\left\{ \Rightarrow \text{------} = 1, \text{ as } \mathrm{char}(C) = p \right.$

$\chi(\varpi) \in C^\times$, $\mathbb{F}_q^\times \longrightarrow C^\times$.

Generalise (*):

p-gp. lemma:

V smooth rep. of pro-p gp. H. (over C)

$\Rightarrow V^H \neq 0$.

Idea:

reduce
- H fin. p-gp.
- $C = \mathbb{F}_p$.
- V f.d.

$\rho_V : H \longrightarrow GL_d(\mathbb{F}_p)$

image $\subset$ Sylow $\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ //

Cor (notes):

If π is sm. G-rep., $H \underset{\substack{\text{open}\\ \text{pro-}p.}}{\leq} G$,

π adm. $\Longleftrightarrow$ π^H f.-d.

Applications to irreducibility.

Ex. 1: $\Gamma = GL_2(\mathbb{F}_q) \curvearrowright V = \mathrm{Sym}^r(\underbrace{\mathbb{C}^2}_{\text{std. rep.}})$

$\boxed{q = p}$

$\qquad\qquad\qquad\qquad\qquad\qquad // $

$\mathbb{C}[X,Y]_{\deg = r}$

homog. polys.

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(X,Y) = f(aX+cY, bX+dY)$$

If $0 \leq r \leq q-1$, then V irred:

Sketch:

$$\Delta := \begin{pmatrix} 1 & \\ \mathbb{F}_q & 1 \end{pmatrix} \leq \Gamma \text{ (Sylow)}$$

$$\left.\begin{array}{l} ① \ \ V^{\Delta} = C \cdot Y^r \text{ is 1-dim.} \\[1em] ② \ \ \langle \Gamma \cdot V^{\Delta} \rangle = V \end{array}\right\} \text{calculation}$$

(gen. by Γ-action)

If $0 \neq W \subset V$ (Γ-subrep.)

$$\underset{\text{lemma}}{\Longrightarrow} \quad W^{\Delta} \neq 0.$$
$$\qquad \qquad \quad \shortparallel$$
$$\qquad \qquad \quad V^{\Delta} \text{ by } ①.$$

$$\underset{②}{\Longrightarrow} V = \langle \Gamma \cdot V^{\Delta} \rangle \subset W, \text{ so } V = W.$$

//

<u>Cor:</u> The irreps. of Γ over C are $\mathrm{Sym}^r(C^2) \otimes \det^s$

$0 \leq r \leq q-1$
$s \in \mathbb{Z}/(q-1)$.
(total $= q(q-1)$)

"<u>Serre weights</u>".

<u>Ex. 2:</u> $K = GL_2(\mathcal{O}_F)$.

$\triangledown$

K_1 pro-p.

If V <u>irred.</u> sm. rep. of K,

$\underset{\text{Lemma}}{\Longrightarrow} 0 \neq V^{K_1} \subset V$ ... also a K-subrep.

$\Rightarrow V^{K_1} = V$, so V is an irrep. of $K/K_1 \cong \Gamma$, a Serre weight.

<u>Ex. 3</u>: $G = GL_2(F)$

$B = \begin{pmatrix} * & * \\ & * \end{pmatrix} \leq G$ Borel = min. parab.

$B = T \ltimes U$, $T = \begin{pmatrix} * & \\ & * \end{pmatrix}$
$U = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$.

Recall:
$\theta = \theta_1 \otimes \theta_2 : T \longrightarrow \mathbb{C}^\times$ smooth.

$\boxed{\pi}$:= $\underline{\mathrm{Ind}_B^G(\theta)}$ =

$$= \left\{ f: G \to \mathbb{C} \text{ loc. const}: \begin{array}{l} f\left(\begin{pmatrix} \alpha & \beta \\ & \delta \end{pmatrix} g\right) = \theta_1(\alpha)\theta_2(\delta) f(g) \\ \forall \alpha, \delta \in F^\times, \beta \in F, \\ g \in G \end{array} \right\}$$

is an adm. smooth G-rep.

__Thm__ (Barthel–Livné):

If $\theta_1 \neq \theta_2$, then $\mathrm{Ind}_B^G(\theta)$ irred.

__Idea:__ Assume $\theta_1|_{\mathcal{O}_F^\times} \neq \theta_2|_{\mathcal{O}_F^\times}$

$\Downarrow$

① $\pi|_K$ contains a <u>unique</u> irred. K-subrep. V

[use: $(\mathrm{Ind}_B^G \theta)|_K = \mathrm{Ind}_{B\cap K}^K \theta$

+ Frob. rec.]

② $\langle G\cdot V\rangle = \pi$

If $0 \neq \pi' \subset \pi$ (G-subrep.),

then $\pi'|_K$ contains an irred. K-sub

$\underset{①}{\Longrightarrow}$ $V \subset \pi'$

$\underset{②}{\Longrightarrow}$ $\pi = \langle G\cdot V\rangle \subset \pi'$ $\Rightarrow$ $\underline{\pi' = \pi}$ //

Rk: If $\theta_1 = \theta_2$, $\theta_0 := \theta_1 = \theta_1$.

$$\mathrm{Ind}_B^G(\theta_1 \otimes \theta_2) = (\theta_0 \circ \det) \otimes \underbrace{\mathrm{Ind}_B^G(1)}_{\mathcal{C}^\infty(B\backslash G)}.$$

$$0 \to \underset{\text{const.}}{\mathbb{1}} \longrightarrow \mathrm{Ind}_B^G(1) \to \underset{\text{Steinberg.}}{St} \longrightarrow 0$$

Thm. (B–L)

The irred. adm. rep. of $GL_2(F)$:

- $\theta_0 \circ \det$ (1-dim)
- $\mathrm{Ind}_B^G(\theta_1 \otimes \theta_2)$, $\theta_1 \neq \theta_2$
- $St \otimes (\theta_0 \circ \det)$
- "supersingular"

+ and no nontriv. isoms. between them.