AWS 2025: Outline of the course "Representations of p-adic groups"

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A *p-adic group* for us is a topological group $G(F)$, where F is a non-archimedean local field whose residue field has characteristic p, and G is a connected reductive group over F. These p-adic groups include groups like $\mathrm{GL}_n(F), \mathrm{SL}_n(F), \mathrm{SO}_n(F), \mathrm{Sp}_{2n}(F), \ldots$, but also groups of exceptional types G_2, F_4, E_6, E_7 and E_8. This lecture series concerns the study of the category of smooth complex representations of $G(F)$. A representation of $G(F)$, i.e., a group homomorphism from $G(F)$ to the space of linar automorphisms of a complex (often infinite dimensional) vector space V is called smooth if the stabilizer in $G(F)$ of any vector $v \in V$ contains an open subgroup. A basis of compact, open neighborhoods of $\mathrm{GL}_n(F)$ is, for example, given by

$$\mathrm{GL}_n(\mathbb{Z}_p) \supset \mathrm{Id} + p \,\mathrm{Mat}_{n\times n}(\mathbb{Z}_p) \supset \mathrm{Id} + p^2 \,\mathrm{Mat}_{n\times n}(\mathbb{Z}_p) \supset \mathrm{Id} + p^3 \,\mathrm{Mat}_{n\times n}(\mathbb{Z}_p) \supset \ldots.$$

This filtration of a p-adic group by compact, open subgroups was vastly generalized by Moy and Prasad ([MP94, MP96]) building up on the foundational work of Bruhat and Tits ([BT72, BT84], see also [KP23]), and plays a crucial role in the construction and classification of smooth representations of p-adic groups.

The building blocks of all smooth representations of $G(F)$ are called *supercuspidal* representations, and every irreducible representation embeds into the parabolic induction of a supercuspidal representation of a Levi subgroup of G. More precisely, by Bernstein ([Ber84]), the category of smooth representations $\mathcal{R}(G)$ of $G(F)$ decomposes into a product of full subcategories:

$$\mathcal{R}(G) \simeq \prod \mathcal{R}(G)^{[M,\sigma]},$$

where the product is taken over inertial equivalence classes $[M, \sigma]$ of pairs (M, σ) consisting of a Levi subgroup M of G and a supercuspidal representation σ of $M(F)$. Two pairs are inertially equivalent if one can be obtained from the other by conjugation by $g \in G(F)$ and twisting σ by an unramified character of $M(F)$, i.e., a character that is trivial on all compact subgroups. The indecomposable subcategory $\mathcal{R}(G)^{[M,\sigma]}$ is called a Bernstein block and consists of all the smooth representations of $G(F)$ all of whose irreducible subquotients are subrepresentations of $\mathrm{Ind}_{P'(F)}^{G(F)}(\sigma')$ where P' is a parabolic subgroup with Levi subgroup M' and (M', σ') is inertially equivalent to (M, σ).

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Thus, understanding the category of smooth representations of $G(F)$ reduces to understanding supercuspidal representations and understanding the structure of Bernstein blocks. Under minor tameness assumptions, we know how to construct all supercuspidal representations explicitly ([Yu01, Fin21a, Kim07, Fin21b]). This is done via compact induction of finite dimensional representations of compact-mod-center open subgroups, which we will sketch in the lecture.

The structure of the Bernstein blocks can be analyzed via type theory. A pair (K, ρ) consisting of a compact subgroup K of $G(F)$ and an irreducible smooth representation ρ of K is an $[M, \sigma]$-*type* if the following property holds: For every irreducible smooth representation π of $G(F)$ the following are equivalent:

(i) $\pi \in \mathcal{R}(G)^{[M,\sigma]}$,

(ii) ρ is a subrepresentation of the restriction $\pi|_K$ of π to K (i.e., $\mathrm{Hom}_K(\rho, \pi) \neq \{0\}$).

If (K, ρ) is an $[M, \sigma]$-type, then the Bernstein block $\mathcal{R}(G)^{[M,\sigma]}$ is equivalent to the category of right unital $\mathcal{H}(G, K, \rho)$-modules ([BK98]), i.e.,

$$\mathcal{R}(G)^{[M,\sigma]} \simeq \mathrm{Mod} - \mathcal{H}(G, K, \rho),$$

where $\mathcal{H}(G, K, \rho)$ is the Hecke algebra defined as

$$\mathcal{H}(G, K, \rho) = \{f : G(F) \to \mathrm{End}_{\mathbb{C}}(V_\rho) \mid f \text{ is compactly supported and } f(kgk') = \rho(k)f(g)\rho(k') \text{ for all } k, k' \in K, g \in G(F)\},$$

where V_ρ denotes the (finite dimensional complex) vector space of the representation ρ and where the product structure of the algebra is given by a convolution formula.

The structure of these Hecke algebras $\mathcal{H}(G, K, \rho)$ in turn can be described rather explicitly as a semidirect product of a twisted group algebra with an affine Hecke algebra:

$$\mathbb{C}[\Omega(\rho), \mu] \ltimes \mathcal{H}(W(\rho)_{\mathrm{aff}}, q)$$

([AFMOa, AFMOb]). We will explain interesting isomorphisms between these Hecke algebras attached to different Bernstein blocks and sketch applications of these results as part of the lecture series. The projects will be concerned with achieving a better understanding of the parameters appearing in these Hecke algebras.

Overview of the content of the lectures

The course will begin with a brief review of p-adic groups and some of their structures, definitions of smooth representations, parabolic induction and supercuspidal representations. We will introduce the Bernstein decomposition and explain that each block is equivalent to modules over a Hecke algebra, under minor tameness assumptions. We sketch the structure of these Hecke algebras and the "reduction to depth-zero" isomorphism. We will also give a brief introduction to the Moy–Prasad filtration and Bruhat–Tits theory and discuss the construction of supercuspidal representations and the notion of types underlying the Hecke algebras.

Projects

Recently the Hecke algebras attached to Bernstein blocks (under minor tameness assumptions) were explicitly described in [AFMOa, AFMOb]. However, “explicitly” means that the authors provide a recipe for how to calculate the coefficients involved in the algebra structure, but do not actually compute them. In the projects we will try to better understand these coefficients. This includes calculating them explicitly for some examples and trying to prove some important conjectures about their shape, at least in special cases, maybe for some interesting families of groups.

The project will involve studying parahoric subgroups, their quotients by pro-p-unipotent radicals and representations of (disconnected) finite groups of Lie type.

Prerequisites for the projects:

- familiarity with non-archimedean local fields
- some basic knowledge of the representation theory of finite groups
- some familiarity with reductive groups over non-archimedean local fields, at least with the special cases of $\mathrm{GL}_n(\mathbb{Q}_p)$ and $\mathrm{SL}_n(\mathbb{Q}_p)$ and one more group, e.g. orthogonal or symplectic groups, or willingness to learn this by the time the project starts (see [Fin] for an overview, classical references for reductive groups include [Bor91, Hum75, Spr09], a more modern treatment can be found in [Con17a, Con17b])
- familiarity with or willingness to learn before the start of the project work about parahoric subgroups, their stabilizers and unipotent radicals, and some Bruhat–Tits theory, at least as described in [Fin] (for a detailed account of the theory see [KP23])
- willingness to work in a team, promising to treat everyone with respect and support each other and, of course, to follow the code of conduct

Disclaimer: The proposed projects are open questions and progress might lead to a research paper. However, the project leader and the project assistant do not commit to writing or supporting a joint paper after the conclusion of the Arizona Winter School.

Suggested reading and background material

- [Fin] will form the basis for the course notes and provides further references to the relevant literature. Students interested in the project might find Section 2 and 3 particularly helpful. They also need to learn about Hecke algebras (which are not yet part of the notes). A more basic overlapping introduction to the area that provides a few more details on some topics (and less on others) is [Fin23].
- [AFMOa, AFMOb] are research papers on the topic of Hecke algebras, an overview of which will be added to the course notes. Some sections of these papers will be required for the projects.

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